MATH 20A: Lecture 14

10/26/15
We have developed the differential calculus in order to solve a geometric problem: calculate the slope of the tangent line to the graph of a given function $f(x)$ at a given point $(x_0, f(x_0))$. 

\[ \text{slope} = f'(x_0) \]
• However, there are many natural curves in the plane which are not graphs of functions, e.g. the unit circle

\[ U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}. \]

• How do we solve the tangent problem for curves which aren’t graphs of functions?
For the unit circle, we can realize that $\mathcal{U}$ is made up of the graphs of two functions glued together:

$$x^2 + y^2 = 1 \iff y^2 = 1 - x^2 \iff y = \sqrt{1 - x^2} \text{ or } y = -\sqrt{1 - x^2}.$$

That is, if $f_1(x) = \sqrt{1 - x^2}$ and $f_2(x) = -\sqrt{1 - x^2}$, then $\mathcal{U} = \{(x, f_1(x))\} \cup \{(x, f_2(x))\}$. 
• So, for example, if we want to find the slope of the tangent line through \((\frac{3}{5}, \frac{4}{5})\), we can compute the derivative of \(f_1(x)\) at the point \((\frac{3}{5}, \frac{4}{5})\):

\[
f_1'(x) = \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x)
= -x (1-x^2)^{-\frac{1}{2}}
\]

\[
f_1'(\frac{3}{5}) = -\frac{3}{5} (1-\frac{9}{25})^{-\frac{1}{2}}
= -\frac{3}{5} \left(\frac{16}{25}\right)^{-\frac{1}{2}}
= -\frac{3}{5} \cdot \frac{5}{4}
= -\frac{3}{4}
\]
If, on the other hand, we want to find the slope of the tangent line to the unit circle at the point \((\frac{3}{5}, -\frac{4}{5})\), we’d instead differentiate the function \(f_2(x)\):

\[
f_2(x) = -\left(1 - x^2\right)^{1/2}
\]

\[
f_2'(x) = -\frac{1}{2} \left(1 - x^2\right)^{-1/2} \cdot (2x)
= x \left(1 - x^2\right)^{-1/2}
\]

\[
f_2'(\frac{3}{5}) = \frac{3}{5} \left(1 - \frac{9}{25}\right)^{-1/2}
= \frac{3}{5} \left(\frac{16}{25}\right)^{-1/2}
= \frac{3}{5} \cdot \frac{5}{4}
= \frac{3}{4}.
\]
• However, there's a more uniform way to solve this problem—"implicit differentiation."

• Simply write down the defining equation of the unit circle and hit both sides with \(\frac{d}{dx}\), treating \(y\) as an implicitly (rather than explicitly) defined function of \(x\):

\[
\begin{align*}
X^2 + y^2 &= 1 \implies \frac{d}{dx} (x^2 + y^2) &= \frac{d}{dx} (1) \\
&\implies \frac{d}{dx} x^2 + \frac{d}{dx} y^2 = 0 \\
&\implies 2x + 2y \frac{dy}{dx} = 0. \\
&\implies \frac{dy}{dx} = -\frac{x}{y}.
\end{align*}
\]

• Plugging in the coordinates of our two points of interest, \((\frac{3}{5}, \frac{4}{5})\) and \((\frac{3}{5}, -\frac{4}{5})\), we get back the same answers as before in a much quicker fashion:

\[
\begin{align*}
\left. \frac{dy}{dx} \right|_{(x,y)=(\frac{3}{5}, \frac{4}{5})} &= -\frac{3/5}{4/5} = -\frac{3}{4} \\
\left. \frac{dy}{dx} \right|_{(x,y)=(\frac{3}{5}, -\frac{4}{5})} &= -\frac{3/5}{-4/5} = \frac{3}{4}.
\end{align*}
\]
Example: Calculate the slope of the tangent line to the curve

\[ C = \{ (x, y) : y \cos(y + x + x^2) = x^3 \} \]

at the point \( P = (0, \frac{5\pi}{2}) \).

Solution: Use implicit differentiation:

\[
\frac{d}{dx} (y \cos(y + x + x^2)) = \frac{d}{dx} x^3
\]

\[
\Rightarrow \frac{dy}{dx} \cos(y + x + x^2) + y \frac{d}{dx} \cos(y + x + x^2) = 3x^2
\]

\[
\Rightarrow \frac{dy}{dx} \cos(y + x + x^2) - y \sin(y + x + x^2) \left( \frac{dy}{dx} + 1 + 2x \right) = 3x^2.
\]

Now plug in the coordinates of \( P \):

\[
\frac{dy}{dx} \bigg|_P \cos\left(\frac{5\pi}{2}\right) - \frac{5\pi}{2} \sin\left(\frac{5\pi}{2}\right) \left( \frac{dy}{dx} \bigg|_P + 1 \right) = 0
\]

\[
\Rightarrow \frac{dy}{dx} \bigg|_P \cos\left(\frac{\pi}{2}\right) - \frac{5\pi}{2} \sin\left(\frac{\pi}{2}\right) \left( \frac{dy}{dx} \bigg|_P + 1 \right) = 0
\]

\[
\Rightarrow \frac{dy}{dx} \bigg|_P = -1.
\]
Example: Find the equation of the tangent line to the curve \( C = \{ (x,y) : xy + x^2y^2 = 6 \} \) at the point \( P = (2,1) \).

Solution: Use implicit differentiation:

\[
\frac{d}{dx}(xy + x^2y^2) = \frac{d}{dx}(6) \quad \Rightarrow \quad y + x \frac{dy}{dx} + 2xy^2 + x^2 2y \frac{dy}{dx} = 0
\]

\[
\Rightarrow (2x^2y + x) \frac{dy}{dx} = -2xy^2 - y
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{-2xy^2 - y}{2x^2y + x}
\]

Thus

\[
\left. \frac{dy}{dx} \right|_{(2,1)} = \frac{-2 - 2}{10} = -\frac{2}{5}
\]

So now we just need to find the equation of the line through \((2,1)\) with slope \(-\frac{2}{5}\):

\[
\frac{y - 1}{x - 2} = -\frac{2}{5} \quad \Rightarrow \quad y - 1 = -\frac{2}{5}(x - 2) \quad \Rightarrow \quad y = -\frac{2}{5}x + \frac{4}{5} + 1 \quad \Rightarrow \quad y = -\frac{2}{5}x + \frac{9}{5}
\]
Example: The curve $F = \{(x,y): x^3 + y^3 = 3xy\}$ is called the "Folium of Descartes." Find all points on $F$ where the tangent line is horizontal.

Solution: Use implicit differentiation: 

$$3x^2 + 3y^2 \frac{dy}{dx} = 3y + 3x \frac{dy}{dx}$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

Thus $\frac{dy}{dx} = 0$ if and only if $y = x^2$. Plugging this into the equation which defines $F$ yields

$$x^3 + x^6 = 3x^3 \iff x^6 - 2x^3 = 0$$

$$\iff x^3 (x^3 - 2) = 0$$

$$\iff x = 0 \text{ or } x = \sqrt[3]{2} \approx 1.26$$

Thus, there are two points on the Folium at which a horizontal tangent occurs: $(0,0)$ and $(2^{1/3}, 2^{1/6})$. 
Example: The curve $L = \{(x,y) : (x^2+y^2)^2 = 2(x^2-y^2)\}$ is called the "lemniscate of Bernoulli." Find a formula for the slope of the tangent line to $L$ at $P=(x,y)$.

Solution: Use implicit differentiation:

$$2(x^2+y^2)(2x+2y \frac{dy}{dx}) = 2(2x-2y \frac{dy}{dx})$$

$$\iff (x^2+y^2)(x+y \frac{dy}{dx}) = (x-y \frac{dy}{dx})$$

$$\iff x^3 + x^2y \frac{dy}{dx} + x y^2 + y^3 \frac{dy}{dx} = x - y \frac{dy}{dx}$$

$$\iff (y^3 + x^2y + y \frac{dy}{dx}) \frac{dy}{dx} = -x^3 - xy^2 + x$$

$$\iff \frac{dy}{dx} = \frac{-x^3 - xy^2 + x}{y^3 + x^2y + y}.$$