Mean Value Theorem: Suppose that $f(x)$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$
Note that the hypothesis of differentiability is required; continuity won’t do it.
• Linear approximation: if $b$ is close to $a$, then
  \[ f(b) \approx f(a) + f'(a)(b-a). \]

• Mean Value Theorem: there exists $c$ between $a$ and $b$ such that
  \[ f'(c) = \frac{f(b)-f(a)}{b-a} \iff f(b) = f(a) + f'(c)(b-a). \]

**Proposition:** If $f(x)$ is differentiable on $(a,b)$, and $f'(x) = 0$ for all $x \in (a,b)$, then $f(x)$ is a constant function.

**Proof:** For any $a, b \in (a,b)$, MVT gives $c \in (a, b)$ such that $f(b) = f(a) + f'(c)(b-a) = f(a)$. 
Proposition: If \( f(x) \) is differentiable on \((a,b)\) and \( f'(x) > 0 \) on \((a,b)\), then \( f(x) \) is increasing on \((a,b)\). If \( f'(x) < 0 \) on \((a,b)\), then \( f(x) \) is decreasing on \((a,b)\).

Proof: • Let \( a < b \) be any two points in the interval \((a,b)\).

• By MVT, there exists \( c \in (a,b) \) such that

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

• If \( f'(c) > 0 \),

\[
\frac{f(b) - f(a)}{b - a} = f'(c) > 0 \implies f(b) > f(a).
\]

• If \( f'(c) < 0 \), then

\[
\frac{f(b) - f(a)}{b - a} = f'(c) < 0 \implies f(b) < f(a).
\]
First Derivative Test for Critical Points

Let $f$ be differentiable on $(a, b)$ and let $c \in (a, b)$ be a critical point of $f(x)$, i.e. a point such that $f'(c) = 0$. Is $c$ a local max, a local min, or neither?

**Case I:** If $f'(x)$ changes sign from positive to negative at $c$, then $c$ is a local max:

$$f'(c) = 0$$

$$f'(x) > 0 \implies f'(x) < 0$$

**Case II:** If $f'(x)$ changes sign from negative to positive at $c$, then $c$ is a local min:

$$f'(c) = 0$$

$$f'(x) < 0 \implies f'(x) > 0$$

**Case III:** If $f'(x)$ doesn’t change sign at $c$, then $c$ is neither a local max nor a local min:
Example: Analyze the critical points of \( f(x) = x^3 - x \).

Solution: • Since \( f'(x) = 3x^2 - 1 \), the critical points of \( f(x) \) are

\[
c_1 = -\frac{1}{\sqrt{3}} \quad \text{and} \quad c_2 = \frac{1}{\sqrt{3}}.
\]

• The derivative itself, \( f'(x) \), is an upward-opening parabola which crosses the \( x \)-axis at \( c_1 \) and \( c_2 \):

![Graph of a parabola with critical points at \( c_1 \) and \( c_2 \).]

• By the First Derivative Test, \( c_1 \) is a local max, and \( c_2 \) is a local min.
Example: Sketch the graph of the function \( f(x) = x^3 - x \).

Solution: We already know that the critical points of \( f(x) \) are \( c_1 = -\frac{1}{\sqrt{3}} \) and \( c_2 = \frac{1}{\sqrt{3}} \), with \( c_1 \) a local max and \( c_2 \) a local min. Since \( c_1 = -c_2 \) and \( f(x) \) is an odd function, \( f(c_1) = -f(c_2) \).
• The zeros of \( f(x) = x^3 - x = x(x-1)(x+1) \) are \(-1, 0, \text{ and } 1\).
• Since \( \lim_{x \to -\infty} (x^3 - x) = -\infty \) and \( \lim_{x \to \infty} (x^3 - x) = +\infty \), \( f(x) \) decreases without bound left of \( x = -1 \), and increases without bound right of \( x = 1 \): 

\[
\int f(x) = x^3 - x
\]
• Certain special cases of the Mean Value Theorem were stated by Parameshvara (circa 1400).

• The modern form of the MVT was obtained by Cauchy (circa 1800), who actually proved a more general statement involving two functions.
Cauchy’s Mean Value Theorem: If \( f(x) \) and \( g(x) \) are continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists \( c \in (a, b) \) such that

\[
(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).
\]

Proof: • Define a new function \( h(x) \) by

\[
h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x).
\]

• The function \( h(x) \) inherits the continuity and differentiability properties of \( f(x) \) and \( g(x) \). Moreover,

\[
h(a) = h(b) \Rightarrow \frac{h(b) - h(a)}{b - a} = 0.
\]

• Original MVT then implies existence of \( c \in (a, b) \) so that

\[
h'(c) = (f(b) - f(a))g'(x) - (f(b) - f(a))g'(x) = 0.
\]
Cauchy’s MVT says that \( \exists c \in (a,b) \) such that
\[
\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)};
\]
this is where l’Hôpital’s Rule comes from.

Geometrically, Cauchy’s MVT is a statement about secants and tangents of the parameterized curve \( (x(t), y(t)) = (f(t), g(t)) \):