Theorem (l’Hôpital’s Rule): Suppose that \( f(x) \) and \( g(x) \) are differentiable on an open interval containing \( x_0 \), and that

\[
f(x_0) = g(x_0) = 0.\]

Then,

\[
\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \quad \Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.
\]

Proof: l’Hôpital’s rule can be deduced from the Cauchy Mean Value Theorem, which we proved in Lecture 19.
Example: Evaluate \( \lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} \).

Solution: Since \( f(x) = x^3 + x - 2 \) and \( g(x) = x^2 - 3x + 2 \) are differentiable (on all of \( \mathbb{R} \)) and \( f(1) = g(1) = 0 \), l'Hôpital's rule applies.
Example: Can \( \lim_{{x \to 1}} \frac{3x^2 + 1}{2x - 3} \) be evaluated using l'Hôpital's rule?

Solution: No. While \( f(x) = 3x^2 + 1 \) and \( g(x) = 2x - 3 \) are certainly differentiable functions, it is not the case that \( f(1) = g(1) = 0 \). Indeed, if you try to use l'Hôpital's rule to evaluate this limit, you get

\[
\lim_{{x \to 1}} \frac{3x^2 + 1}{2x - 3} = \lim_{{x \to 1}} \frac{6x}{2} = 3,
\]

which is wrong (we know from the previous example that the limit is \(-4\)).
• The previous example was not intended to scare you away from iterating l'Hôpital's rule — only to ensure that you're careful when doing so.

Example: Evaluate \( \lim_{x \to 0} \frac{e^x - x - 1}{\cos(x) - 1} \).

Solution:

• Since \( f(x) = e^x - x - 1 \) and \( g(x) = \cos(x) - 1 \) are differentiable functions satisfying \( f(0) = g(0) = 0 \), l'Hôpital's rule applies: if it is the case that \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) exists and equals \( L \), then it is also the case that \( \lim_{x \to 0} \frac{f(x)}{g(x)} \) exists and equals \( L \).

• However, since \( f'(x) = e^x - 1 \) and \( g'(x) = -\sin(x) \), we have \( f'(0) = g'(0) = 0 \), so \( \lim_{x \to 0} \frac{f'(x)}{g'(x)} \) cannot be evaluated directly. But l'Hôpital's rule applies.

• We compute \( f''(x) = e^x \), \( g''(x) = -\cos(x) \), so

\[
\lim_{x \to 0} \frac{f''(x)}{g''(x)} = -1 \Rightarrow \lim_{x \to 0} \frac{f'(x)}{g'(x)} = -1 \Rightarrow \lim_{x \to 0} \frac{f(x)}{g(x)} = -1.
\]
Theorem (l'Hôpital's Rule): Suppose that \( f(x) \) and \( g(x) \) are differentiable on an open interval containing \( x_0 \), and that

\[
\begin{align*}
\lim_{x \to x_0} f(x) &= g(x) = 0.
\end{align*}
\]

Then,

\[
\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \quad \Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.
\]

Remark: This remains valid in the case \( L = \infty \). That is, if \( f(x) \) and \( g(x) \) satisfy the above hypotheses and

\[
\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = \infty,
\]

then it is also the case that \( \lim_{x \to x_0} \frac{f(x)}{g(x)} = \infty \). It is also valid for one-sided limits.
Example: Show that $\lim_{x \to 0^+} \frac{x}{1 - \cos(x)} = \infty$.

Solution: Both $f(x) = x$ and $g(x) = 1 - \cos(x)$ are differentiable.

- Since $f(0) = g(0) = 0$, l'Hôpital’s rule applies.

- We have $f'(x) = 1$, $g'(x) = \sin(x)$, so that

$$\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0^+} \frac{1}{\sin(x)} = \infty.$$

- Hence, by l'Hôpital’s rule,

$$\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} \frac{x}{1 - \cos(x)} = \infty.$$
Theorem (\(\frac{\infty}{\infty}\) - l'Hôpital's rule)

Suppose that \(f(x)\) and \(g(x)\) are differentiable in a neighbourhood of \(x_0\), and

\[
\lim_{x \to x_0} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to x_0} g(x) = \pm \infty.
\]

Then

\[
\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L \quad \Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = L.
\]

This remains true for one-sided limits, or if \(L\) is infinite.
Example: Evaluate \( \lim_{x \to 0^+} x \ln(x) \).

Solution:  
- Note that we can write \( x \ln(x) = \frac{f(x)}{g(x)} \) with \( f(x) = \ln(x) \) and \( g(x) = \frac{1}{x} \).

- Both \( f(x) \) and \( g(x) \) are differentiable on \((0, \infty)\).

- Since \( \lim_{x \to 0^+} f(x) = -\infty \) and \( \lim_{x \to 0^+} g(x) = +\infty \), \( l'Hôpital's \) rule applies.

- We have \( f'(x) = \frac{1}{x} \) and \( g'(x) = -\frac{1}{x^2} \), so \( \frac{f'(x)}{g'(x)} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x \), and

\[
\lim_{x \to 0^+} \frac{f(x)}{g(x)} = \lim_{x \to 0^+} (-x) = 0
\]

- We conclude that \( \lim_{x \to 0^+} x \ln(x) = 0 \).
Theorem (l’Hôpital’s Rule for Limits at Infinity)

Suppose that \( f(x) \) and \( g(x) \) are differentiable on an interval \((a, \infty)\). Then

\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to \infty} \frac{f(x)}{g(x)} = L.
\]

The result holds for \( L \) infinite, and for one-sided limits.

Similarly, if \( f(x) \) and \( g(x) \) are differentiable on an interval \((-\infty, a)\), then

\[
\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = L \implies \lim_{x \to -\infty} \frac{f(x)}{g(x)} = L.
\]
Example: A math professor tells you that the number of prime numbers less than a given number \( n \) is approximately \( \frac{n}{\ln(n)} \), if \( n \) is large. For example, the number of primes less than 1,000,000 is approximately \( \frac{1,000,000}{6} \). Is this claim compatible with what you know about the primes?
Solution: Presumably, you know Euclid’s theorem: there are infinitely many prime numbers.

Thus, in order for the professor’s claim to be at least plausible, it must be the case that

$$\lim_{x \to \infty} \frac{x}{\ln(x)} = \infty.$$ 

Since $f(x) = x$ and $g(x) = \ln(x)$ are differentiable on $(0, \infty)$, l’Hôpital’s rule applies.

We have $f'(x) = 1$, $g'(x) = \frac{1}{x}$, so

$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{1}{\frac{1}{x}} = \lim_{x \to 1} x = \infty.$$ 

Therefore $\lim_{x \to \infty} \frac{x}{\ln(x)} = \infty$, by l’Hôpital’s rule.