MATH 20A: Lecture 21

13/11/15

§4.4, 4.6
• Reminder: the critical points of a function $f(x)$ are the points where the derivative vanishes:

$$c \text{ is a critical point of } f(x) \iff f'(c) = 0.$$  

• Critical points matter because they are the points at which the slope of the tangent line to $f(x)$ is zero, which is a necessary condition for local extrema:

$$f'(c) = 0$$

• Caution: Not every critical point is a local extremum,
First Derivative Test: Suppose that $c$ is a critical point of $f(x)$.

1. If $f'(x)$ changes sign from positive to negative at $c$, then $c$ is a local max for $f(x)$.

2. If $f'(x)$ changes sign from $-$ to $+$ at $c$, then $c$ is a local min for $f(x)$.

3. If $f'(x)$ does not change sign at $c$, then $c$ is neither a local max nor a local min.
Second Derivative Test: Suppose that $c$ is a critical point of $f(x)$.

(1) If $f''(c) < 0$, then $c$ is a local max of $f(x)$.

(2) If $f''(c) > 0$, then $c$ is a local min of $f(x)$. 
Example: Locate the critical points of \( f(x) = x^3 - x \), and determine whether each is a local max, a local min, or neither.

Solution: • The critical points of \( f(x) = x^3 - x \) are the zeroes of the derivative:

\[
f'(x) = 0 \iff 3x^2 - 1 = 1 \iff x = -\frac{1}{\sqrt{3}} \text{ or } x = \frac{1}{\sqrt{3}}.
\]

• The second derivative of \( f(x) \) is \( f''(x) = 6x \).

• Plug the critical points into \( f''(x) \):

\[
f''(-\frac{1}{\sqrt{3}}) = -\frac{6}{\sqrt{3}} < 0 \quad f''(\frac{1}{\sqrt{3}}) = \frac{6}{\sqrt{3}} > 0.
\]

• By SDT, \( -\frac{1}{\sqrt{3}} \) is a local max, \( \frac{1}{\sqrt{3}} \) is a local min.
\[ f(x) = x^3 - x \]
Example: Locate and classify the critical points of \( f(x) = x^5 - 5x^4 \).

Solution: The critical points of \( f(x) = x^5 - 5x^4 \) are the zeros of the derivative:
\[
f'(x) = 0 \iff 5x^4 - 20x^3 = 0 \iff 5x^3(x - 4) \iff x = 0 \text{ or } x = 4.
\]

We have \( f''(x) = 20x^3 - 60x^2 \). So \( f''(0) = 0 \) and \( f''(4) = 20 \cdot 81 - 60 \cdot 16 > 0 \); 
\( x = 4 \) is a local min, but SDT inconclusive at \( x = 0 \).

Revert to FDT. Since \( f'(-1) = 25 \) and \( f'(1) = -15 \), \( f'(x) \) changes sign from 
+ to - at \( x = 0 \), so \( x = 0 \) is a local max.
Example: Locate the critical points of $f(x) = x^3$, and determine whether each is a local max, a local min, or neither.

Solution: • The critical points of $f(x) = x^3$ are the zeroes of the derivative:

$$f'(x) = 0 \iff 3x^2 = 0 \iff x=0.$$ 

• Try the SDT: $f''(x) = 6x$, so $f''(0) = 0$. Inconclusive.

• However, it's clear that $f'(x) = 3x^2$ is always positive (never changes sign), so the FDT says that $x=0$ is neither a local min nor a local max.
• Still, it's clear from the (presumably familiar) graph of \( f(x) = x^3 \) that something significant is going on at \( x=0 \): the graph switches from concave to convex.

• Terminology: the critical point \( x=0 \) is an inflection point of \( f(x) = x^3 \).
Refined Second Derivative Test: Let \( c \) be a critical point of \( f(x) \).

1) If \( f''(c) > 0 \), then \( c \) is a local min of \( f(x) \).

2) If \( f''(c) < 0 \), then \( c \) is a local max of \( f(x) \).

3) If \( f''(c) = 0 \), the test is inconclusive. However, if \( f''(x) \) changes sign at \( x=c \), then \( c \) is not a local extremum of \( f(x) \) but rather an inflection point of \( f(x) \).
Example: Sketch the graph of \( f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3 \).

Solution: • Start by finding the critical points. We have \( f'(x) = x^2 - x - 2 = (x+1)(x-2) \), so the critical points are \( x = -1 \) and \( x = 2 \).

• Next, we classify the critical points. We have \( f''(x) = 2x - 1 \), so \( f''(-1) = -3 < 0 \) and \( f''(2) = 3 > 0 \), so that \( x = -1 \) is a local max and \( x = 2 \) is a local min.
Also note that $f''(x) = 2x - 1$ changes sign from $-$ to $+$ at $x = \frac{1}{2}$, so this is where the graph of $f(x)$ is concave for $x < \frac{1}{2}$ and convex for $x > \frac{1}{2}$, with $x = \frac{1}{2}$ itself being the unique inflection point.

Finally, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = +\infty$ because $f(x)$ is a cubic polynomial.
Example: Sketch the graph of $f(x) = xe^x$.

Solution: 
- Start by locating the critical points. Since $f'(x) = e^x + xe^x = (x+1)e^x$, the only critical point is $x = -1$.
- We have $f''(x) = e^x + e^x + xe^x = (x+2)e^x$, so $f''(-1) = (-1+2)e^{-1} > 0$. Thus $x = -1$ is a local min, by the SDT.
- We also see that $f''(x) < 0$ for $x < -2$, so that $f(x)$ is concave for $x < -2$, whereas $f(x)$ is convex for $x > -2$.
- Note that $f(x) = 0$ only at $x = 0$ (x-intercept).
- Obviously $\lim_{x \to \infty} xe^x = \infty$, whereas $\lim_{x \to \infty} xe^x = \lim_{x \to \infty} e^{-x} = -\lim_{x \to \infty} e^{-x} = 0$ by l'Hôpital's rule.
\( f(x) = xe^x \).