**Problem 1:** Determine the equation of the tangent line to the graph of a given function $f(x)$ at a given point $(x_0, f(x_0))$.

**Problem 2:** Determine the area of the region under the graph of a given function $f(x)$, and lying between two given numbers $a < b$. 

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**Math 20A**

**Math 20B**
• Though Problem 1 and Problem 2 look quite different, they have something important in common: both are solved using limits.

• We now know how to solve the tangent problem for a large variety of functions.

• The area problem is harder.

• Today, we’ll see an ingenious solution to the area problem for the power functions, $f(x) = x^p$, due to Pierre de Fermat.
We want to calculate the area $A$ bounded by the graph of $f(x) = x^p$, the vertical lines $x = a$ and $x = b$, and the horizontal axis.
• We can approach this computation by picking a large positive number \( N \), and looking at the numbers \( t_0, t_1, \ldots, t_N \) defined by

\[
\begin{align*}
t_0 &= a, \\
t_1 &= a + \frac{b-a}{N}, \\
t_2 &= a + 2 \frac{b-a}{N}, \\
\vdots \\
t_N &= a + N \frac{b-a}{N} = b.
\end{align*}
\]

• Thus \( t_0, t_1, \ldots, t_N \) is an arithmetic progression with initial term \( t_0 = a \) and constant difference \( d = t_{i+1} - t_i = \frac{b-a}{N} \).
We can overestimate $A$ by computing the cumulative area $A_N$ of $N$ rectangles:

$$f(x) = x^p$$

We have $A = \lim_{N \to \infty} A_N$, so to calculate $A$ we just need to compute this limit.
• We have

\[ A_N = \sum_{i=1}^{N} (t_i - t_{i-1}) t_i^p = \sum_{i=1}^{N} \left( \frac{b-a}{N} \right) \left( a + i \frac{b-a}{N} \right)^p = \frac{b-a}{N} \sum_{i=1}^{N} \left( a + i \frac{b-a}{N} \right)^p = ? \]

• It’s not clear how to compute \( \lim_{N \to \infty} A_N \) from this formula.
• Pick a positive number $N$, and define numbers $t_0, t_1, \ldots, t_N$ by

$$t_i = a \left( \frac{b}{a} \right)^i, \quad i = 0, 1, \ldots, N.$$ 

• Thus, $t_0 = a$, $t_N = b$, and $\frac{t_{i+1}}{t_i} = \left( \frac{b}{a} \right)^N$.

• The sequence $t_0, t_1, \ldots, t_N$ is a geometric progression with initial term $t_0 = a$ and constant ratio $r = \left( \frac{b}{a} \right)^N$,

$$a, ar, ar^2, \ldots, ar^N$$

$$t_0, t_1, t_2, \ldots, t_N$$
• We can overestimate the area we want to compute by adding up the areas of \( N \) rectangles:

\[
f(x) = x^p
\]
The formula for the sum of the areas of these rectangles is:

\[ A_N = \sum_{i=1}^{N} (t_i - t_{i-1}) f_i \]

\[ = \sum_{i=1}^{N} (ar^i - ar^{i-1}) \left( \frac{p}{r} \right) \]

\[ = a^{p+1} \sum_{i=1}^{N} (r^i - r^{i-1}) r^{pi} \]

\[ = a^{p+1} (r-1) \sum_{i=1}^{N} r^{(p+1)i-1} \]

\[ = a^{p+1} \left( \frac{1}{1-r} \right) \sum_{i=1}^{N} r^{(p+1)i} \]

\[ = a^{p+1} \left( \frac{1}{1-r} \right) r^{p+1} \sum_{i=0}^{N-1} r^{(p+1)i} \]
Summing the geometric series, we get

\[
A_n = a^{p+1} \left(1 - \frac{1}{r}\right) r^{p+1} \frac{1 - r^{(p+1)N}}{1 - r^{p+1}}
\]

\[
= a^{p+1} (r-1) r^p \frac{|-r^{(p+1)N}|}{1 - r^{p+1}}
\]

\[
= a^{p+1} \left(1 - r^{(p+1)N}\right) r^p \frac{r-1}{1 - r^{p+1}}
\]

\[
= a^{p+1} \left(r^{(p+1)N} - 1\right) r^p \frac{1}{1 - r^{p+1}}
\]

\[
= a^{p+1} \left(r^{(p+1)N} - 1\right) r^p \frac{1}{1 + r + r^2 + \ldots + r^p}
\]
• Now it's time to remember that \( r = \left(\frac{b}{a}\right)^{\frac{1}{n}} \):

\[
A_n = a^{p+1} \left( r^{(p+1)N} - 1 \right) r^p \frac{1}{1 + r + r^2 + \ldots + r^p} \\
= a^{p+1} \left( \frac{b^{p+1}}{a^{p+1}} - 1 \right) \left( \frac{b^p}{a^p} \right)^{\frac{1}{n}} \frac{1}{1 + \left( \frac{b}{a} \right)^{\frac{1}{n}} + \left( \frac{b^2}{a^2} \right)^{\frac{1}{n}} + \ldots + \left( \frac{b^p}{a^p} \right)^{\frac{1}{n}}} \\
= \left( b^{p+1} - a^{p+1} \right) \left( \frac{b^p}{a^p} \right)^{\frac{1}{n}} \frac{1}{1 + \left( \frac{b}{a} \right)^{\frac{1}{n}} + \left( \frac{b^2}{a^2} \right)^{\frac{1}{n}} + \ldots + \left( \frac{b^p}{a^p} \right)^{\frac{1}{n}}}
\]

• From this formula, we get

\[
\lim_{N \to \infty} A_n = \frac{b^{p+1} - a^{p+1}}{p+1}
\]
Theorem (Fermat): The area $A$ bounded by the graph of $f(x) = x^p$, the vertical lines $x=a$ and $x=b$, and the horizontal axis, is

$$A = \frac{b^{p+1} - a^{p+1}}{p+1}$$