1. **2.5 #8** This problem concerns the following graph.

Write down the dimensions of the four fundamental subspaces for this 6 by 4 incidence matrix, and a basis for each subspace. Try to do as much as possible using information from the graph, and as few row reductions as possible.

**Solution:** First, let’s write down the matrix $A$.

$$A = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 \\
-1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1
\end{bmatrix}$$

For the nullspace of $A$: Notice that if we set $Ax = 0$, we would get the conditions $x_2 - x_1 = 0$, $x_3 - x_1 = 0$, $x_3 - x_2 = 0$, $x_4 - x_2 = 0$, $x_4 - x_1 = 0$, and $x_4 - x_3 = 0$. This says that for any vector $x$ in $\text{Null}(A)$, $x_1 = x_2 = x_3 = x_4$. In other words, $x$ must be a multiple of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The dimension of $\text{Null}(A)$ is 1 and $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}$ is a basis.

For the column space of $A$: The three simple loops (loops which don’t contain other loops) give three conditions on the components of the vectors $b$ for which $Ax = b$ has a solution. These three conditions are $b_1 + b_4 = b_5$, $b_3 + b_6 = b_4$, and $b_2 + b_6 = b_5$. The dimension of the column space will be $6 - 3 = 3$ because $b \in \mathbb{R}^6$ and there are 3 conditions on the $b$’s. We can use the conditions to write $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}$ in terms of only three of the $b$’s, say $b_1, b_4, b_6$. So we get that $b_3 = b_4 - b_6$, $b_5 = b_4 - b_6$, and $b_2 = b_5 - b_6 = b_1 + b_4 - b_6$. This says
\[
\begin{bmatrix}
  b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{bmatrix} = \begin{bmatrix}
  b_1 \\
b_1 + b_4 - b_6 \\
b_4 - b_6 \\
b_4 \\
b_4 - b_6 \\
b_6
\end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b_6 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ so a basis for } \text{Col}(A) \text{ is}
\]
\[
\begin{bmatrix}
  1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0
\end{bmatrix}, \begin{bmatrix}
  0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0
\end{bmatrix}, \begin{bmatrix}
  0 \\ -1 \\ 0 \\ 0 \\ 1
\end{bmatrix}.
\]

For the left nullspace of \(A\): We can get a basis easily by thinking of distributing current evenly around each simple loop. Since there are three simple loops, \(\dim(\text{Null}(A^T)) = 3\). I will choose the counterclockwise orientation for each loop, then put a 1 in each row corresponding to an edge in the loop, and negate the ones where my orientation goes in the opposite direction of the arrow along the edge. For the top left loop, top right loop, and bottom middle loop, respectively, this gives the basis
\[
\begin{bmatrix}
  1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0
\end{bmatrix}, \begin{bmatrix}
  0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1
\end{bmatrix}, \begin{bmatrix}
  0 \\ 1 \\ 1 \\ 0 \\ 1
\end{bmatrix}.
\]

For the row space of \(A\): To find a basis for the row space, just pick the rows corresponding to edges that form a spanning tree in the graph. Any spanning tree has \(n - 1\) edges, so \(\dim(\text{Row}(A)) = 4 - 1 = 3\). There are many choices of spanning trees. I’ll choose the one formed by edges \(e_1, e_2, \text{ and } e_5\), which says the first, second, and fifth rows of \(A\) form a basis for the row space:
\[
\begin{bmatrix}
  -1 \\ 1 \\ 0 \\ 0
\end{bmatrix}, \begin{bmatrix}
  -1 \\ 0 \\ 1 \\ 0
\end{bmatrix}, \begin{bmatrix}
  -1 \\ 0 \\ 0 \\ 1
\end{bmatrix}.
\]

2. We saw that for connected graphs, a basis for \(\text{Null}(A)\) was the vector of all 1’s. For this disconnected graph, find three linearly independent vectors \(x_1, x_2, \text{ and } x_3\) which are in the nullspace of the associated matrix \(A\). Choose \(x_1, x_2, \text{ and } x_3\) to have components which are all \(0\)’s and \(1\)’s.

Solution: First, let’s write down the matrix \(A\).
Now if we set \(Ax = 0\), we would get the conditions
\[
x_2 - x_1 = 0,\quad x_3 - x_1 = 0,\quad x_2 - x_4 = 0,\quad x_4 - x_3 = 0,\quad x_6 - x_5 = 0,\quad x_7 - x_5 = 0,\quad x_{10} - x_8 = 0,\quad \text{and} \quad x_{10} - x_9 = 0.
\]
This says that for any vector \(x\) in \(\text{Null}(A)\), \(x_1 = x_2 = x_3 = x_4,\) and \(x_5 = x_6 = x_7,\) and \(x_8 = x_9 = x_10.\) So any vector \(x\) in \(\text{Null}(A)\) can be written as a combination of the following three basis vectors for \(\text{Null}(A)\):

\[
x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

These three vectors form a basis for \(\text{Null}(A)\) so they are linearly independent. They meet the requirement that all of the components are 0 and 1. Check for yourself that they multiply against \(A\) to give the zero vector.

3. 2.6 #6 What \(3 \times 3\) matrices represent the transformations that

(a) project every vector onto the \(x-y\) plane?
(b) reflect every vector through the \(x-y\) plane?
(c) rotate the \(x-y\) plane through 90 degrees, leaving the \(z\)-axis alone?
(d) rotate the \(x-y\) plane, then \(x-z\), then \(y-z\), through 90 degrees?
(e) carry out the same rotations, but each one through 180 degrees?

The phrase “rotate the \(x-y\) plane through 90 degrees” means if you’re standing at the point \((0, 0, 1)\) looking down the \(z\)-axis at the origin, rotate the \(x-y\) plane 90 degrees counterclockwise.

**Solution:** For each part, we get the answer by thinking geometrically about what the transformation does to each of the standard basis vectors and write

\[
A = \begin{bmatrix}
T(e_1) & T(e_2) & T(e_3)
\end{bmatrix}.
\]

(a) \(A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\)

(b) \(A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\)
(c) $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$