12. Basis and dimension

Recall two definitions:

**Definition 12.1.** The vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m \) are (linearly) dependent if there are scalars \( x_1, x_2, \ldots, x_m \), not all zero, such that
\[
x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{0}.
\]
We say that the vectors are (linearly) independent if they are not dependent.

Linear independence places a restriction on the number \( n \) of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \). If the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m \) are independent then \( n \leq m \). You cannot have too many independent vectors.

At the other extreme we have:

**Definition 12.2.** We say that the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m \) spans \( \mathbb{R}^m \) if every vector \( \vec{b} \in \mathbb{R}^m \) is a linear combination of \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \).

Vectors which span places a restriction on the number \( n \) of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \). If the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m \) span \( \mathbb{R}^m \) then \( n \geq m \). You cannot have too few vectors which span.

**Definition 12.3.** The vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^m \) are a basis of \( \mathbb{R}^m \) if \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) are both independent and \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) span \( \mathbb{R}^m \).

The dimension of \( \mathbb{R}^m \) is \( n \), the size of a basis.

Since we already observed that if the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) are independent then \( n \leq m \) and if the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) span \( \mathbb{R}^m \) then \( n \geq m \). Therefore we must have \( n = m \). Thus \( \mathbb{R}^m \) has dimension \( m \).

In fact we have:

**Theorem 12.4.** Let \( A \) be an \( n \times n \) matrix.

The columns of \( A \) are a basis of \( \mathbb{R}^n \) if and only if \( A \) is invertible.

**Example 12.5.** Let \( I_n \) be the identity matrix. Then \( I_n \) is invertible. The columns of \( A \) are
\[
\vec{e}_1 = (1, 0, \ldots, 0), \quad \vec{e}_2 = (0, 1, \ldots, 0) \quad \text{and} \quad \vec{e}_n = (0, 0, \ldots, 1)
\]
a basis of \( \mathbb{R}^n \), called the standard basis.

**Example 12.6.** Consider the vectors \( \vec{v}_1 = (1, 1) \) and \( \vec{v}_2 = (1, -1) \).

We make a matrix with these columns:
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

The determinant is
\[
ad - bc = 1 \cdot -1 - 1 \cdot 1 = -2 \neq 0.
\]
This matrix is invertible. The vectors $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ are a basis of $\mathbb{R}^2$.

In fact it is not hard to see this directly. $\vec{v}_1$ and $\vec{v}_2$ are not parallel, so they are independent. Two independent vectors in $\mathbb{R}^2$ always span. One can see this both algebraically and geometrically. Algebraically, if two vectors in the plane are independent then the homogeneous equation $A\vec{x} = \vec{0}$ has only one solution, the obvious solution $\vec{x} = (0, 0)$. In this case $A$ must have two pivots and so there are no rows of zeroes. But then the equation $A\vec{x} = \vec{b}$ is always consistent and the two vectors $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$. 