22. 2ND MIDTERM REVIEW

Let’s suppose we start with an \( m \times n \) matrix \( A \) and we consider the problem of trying to solve a system of linear equations:

\[
A \vec{x} = \vec{b}.
\]

There are two natural questions:

1. For which \( \vec{b} \in \mathbb{R}^m \) is there a solution?
2. If for some \( \vec{b} \) there is a solution \( \vec{x} \in \mathbb{R}^n \), how many solutions can we find?

If \( A \vec{x} = \vec{b} \) then \( \vec{b} \) is a linear combination of the columns \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) of \( A \):

\[
x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n = \vec{b}.
\]

This is the answer to question #1.

Note that if \( \vec{x}_p \) is a solution to \( A \vec{x} = \vec{b} \) then the set of all solutions has the form \( \vec{x}_p + \vec{x}_h \), where \( \vec{x}_h \) is any solution to the homogeneous:

\[
A \vec{x} = \vec{0}.
\]

So if the equation \( A \vec{x} = \vec{b} \) has one solution then it has as many solutions as the homogeneous. This is the answer to question #2.

Both the linear span and the solutions to the homogeneous are examples of linear subspaces:

\( H \subset \mathbb{R}^n \) is a linear subspace if

1. \( \vec{0} \in H \),
2. \( H \) is closed under addition, \( \vec{u} \in H \) and \( \vec{v} \in H \) implies \( \vec{u} + \vec{v} \in H \),
3. \( H \) is closed under scalar multiplication, \( \vec{u} \in H \) and \( \lambda \) a scalar implies \( \lambda \vec{u} \in H \).

The span of the columns is called the column space and the solutions to the homogeneous is called the nullspace.

The most basic question one can ask about a linear subspace is what is the dimension, the size of a basis. A basis is a set of vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \) which are both independent and span.

Rank-nullity says that the rank of \( A \), the dimension of the column space plus the nullity of \( A \), the dimension of the nullspace, is \( n \).

**Example 22.1.** What is a basis for the column space, the row space, the nullspace and what is the rank and nullity of the following matrix:

\[
\begin{pmatrix}
1 & -4 & 9 & -7 \\
-1 & 2 & -4 & 1 \\
5 & -6 & 10 & 7
\end{pmatrix}
\]

\]
We apply Gaussian elimination:
\[
\begin{pmatrix}
1 & -4 & 9 & -7 \\
0 & -2 & 5 & -6 \\
0 & 14 & -35 & 42
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -4 & 9 & -7 \\
0 & 1 & -5/2 & 3 \\
0 & 2 & -5 & 6
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & -4 & 9 & -7 \\
0 & 1 & -5/2 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

There are pivots in the first and second column. The first and second columns of \(A\) are a basis for the column space:
\[
\vec{c}_1 = (1, -1, 5) \quad \text{and} \quad \vec{c}_2 = (-4, 2, -6).
\]
The rank is 2, the dimension of the column space.

There are pivots in the first and second row. The first and second row of the end product of elimination are a basis for the row space:
\[
\vec{r}_1 = (1, -4, 9, -7) \quad \text{and} \quad \vec{r}_2 = (0, 1, -5/2, 3).
\]
Note that the dimension of the row space is the same as the dimension of the column space. This is part of the statement of rank-nullity.

To find the null space we need to solve the homogeneous. We do this by back substitution. \(x\) and \(y\) are basic variables, \(z\) and \(w\) are free variables.
\[
y - 5z/2 + 3w = 0 \quad \text{so that} \quad y = 5z/2 - 3w.
\]
But then
\[
x - 10z + 12w + 9z - 7w = 0 \quad \text{so that} \quad x = z - 5w.
\]
The general solution is:
\[
(x, y, z, w) = (z - 5w, 5z/2 - 3w, z, w) = z(1, 5/2, 1, 0) + w(-5, -3, 0, 1).
\]
A basis is given by
\[
\vec{n}_1 = (2, 5, 2, 0) \quad \text{and} \quad \vec{n}_2 = (-5, -2, 0, 1).
\]
The nullity is 2. Note that the rank plus the nullity is 2 + 2 = 4, as expected.

**Example 22.2.** Are the vectors \(\vec{v}_1 = (-1, 3, 5, 4), \vec{v}_2 = (2, 4, 2, 2), \vec{v}_3 = (3, 3, 6, 4), \vec{v}_4 = (0, 0, 6, 3)\) a basis of \(\mathbb{R}^4\)?

Let \(A\) be the matrix whose columns are the vectors \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) and \(\vec{v}_4:\)
\[
\begin{pmatrix}
-1 & 2 & 3 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 6 & 6 \\
4 & 2 & 4 & 3
\end{pmatrix}
\]

Then \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) and \(\vec{v}_4\) are a basis if and only if \(A\) invertible. Indeed, \(\vec{v}_1, \vec{v}_2, \vec{v}_3\) and \(\vec{v}_4\) are a basis if and only if
\[
A\vec{x} = \vec{b}
\]

\[2\]
has exactly one solution. This happens if and only if \( A \) is invertible.

To check whether or not \( A \) is invertible we could either apply Gaussian elimination or compute the determinant:

\[
\begin{vmatrix}
-1 & 2 & 3 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 6 & 6 \\
4 & 2 & 4 & 3 \\
\end{vmatrix} = \begin{vmatrix}
-1 & 2 & 3 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 0 & 6 \\
4 & 2 & 1 & 3 \\
\end{vmatrix} = \begin{vmatrix}
-4 & -2 & 0 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 0 & 6 \\
4 & 2 & 1 & 3 \\
\end{vmatrix} = \begin{vmatrix}
2 & 1 & 0 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 0 & 2 \\
4 & 2 & 1 & 1 \\
\end{vmatrix} = 2 \cdot 3 \\
\begin{vmatrix}
2 & 1 & 0 & 0 \\
3 & 4 & 3 & 0 \\
5 & 2 & 0 & 2 \\
-1 & 0 & 1 & -1 \\
\end{vmatrix} = 2 \cdot 3 \\
\begin{vmatrix}
4 & 3 & 0 \\
2 & 0 & 2 \\
0 & 1 & -1 \\
\end{vmatrix} - 1 = 3 \begin{vmatrix}
3 & 3 & 0 \\
0 & 1 & -1 \\
-1 & 1 & -3 \\
\end{vmatrix} \\
= -2 \cdot 3(2 \cdot 2 \begin{vmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
\end{vmatrix} + 3 \begin{vmatrix}
1 & 1 & 0 \\
-1 & 1 & -1 \\
\end{vmatrix} \\
= -2 \cdot 3(2 \cdot 2(-3 + 4) + 3(-2 + 3)) \\
= 2 \cdot 3(2 \cdot 2 + 3) \\
= 42.
\]

Therefore \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) and \( \vec{v}_4 \) are a basis.