Here are a slew of practice problems for the second midterm culled from old midterms:

1. Let

\[
A = \begin{pmatrix}
1 & 3 & 1 \\
1 & 4 & 5 \\
2 & 8 & 11
\end{pmatrix}
\]

Compute \( A^{-1} \).

2. Let

\[
A = \begin{pmatrix}
2 & 3 & 0 \\
1 & 3 & 5 \\
0 & 2 & 1
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
4 & 0 & -1 \\
0 & -3 & 4 \\
0 & 4 & 3
\end{pmatrix}
\]

(a) Compute \( \det A \).
(b) Compute \( \det B \).
(c) Compute \( \det AB \).
(d) Compute \( \det A^T \).
(e) Which of \( A, B, AB \) and \( A^T \) are invertible?

3. Let

\[
A = \begin{pmatrix}
2 & 4 & 6 \\
2 & 5 & 8 \\
-2 & -3 & -4
\end{pmatrix}
\]

(a) Find a basis for \( \text{Col}(A) \).
(b) Find a basis for \( \text{Row}(A) \).
(c) Find a basis for \( \text{Nul}(A) \).

3. (a) If a 7 \times 5 matrix \( A \) has rank 2 find:
(i) \( \dim \text{Nul}(A) \).
(ii) \( \text{rank}(A^T) \).

(b) If the null space of a 4 \times 6 matrix \( A \) is 3-dimensional, what is \( \dim \text{Col}(A) \)?

4. The sets

\[
\mathcal{B} = \{ (-1, 8), (1, -7) \} \quad \text{and} \quad \mathcal{C} = \{ (1, 2), (1, 1) \}
\]

are both bases of \( \mathbb{R}^2 \).

Find the coordinates of the vectors in \( \mathcal{B} \) in terms of the basis \( \mathcal{C} \).
5. Let
\[ A = \begin{pmatrix}
1 & -1 & 2 & 3 & 0 \\
2 & -1 & 4 & 11 & 3 \\
-1 & 3 & -2 & 8 & 4 \\
1 & 1 & 2 & 14 & 4
\end{pmatrix}. \]

(a) Find a basis for \( \text{Col}(A) \).
(b) Find a basis for \( \text{Row}(A) \).
(c) Find a basis for \( \text{Nul}(A) \).

6. Find the determinant of the matrix
\[ \begin{pmatrix}
2 & 3 & -2 & 1 \\
0 & 2 & 5 & 4 \\
0 & -3 & 2 & -3 \\
0 & 1 & 1 & 2
\end{pmatrix}. \]

7. Let \( P_2 \) denote the space of polynomials of degree no greater than 2. Let
\[ W = \{ p \in P_2 | p(-2) = 0 \}. \]

(a) Verify that \( H \) is a linear subspace of \( P_2 \).
(b) Give a careful definition of what is meant by a basis for a vector space.
(c) Find a basis for \( H \). Justify your answer.

8. Let
\[ A = \begin{pmatrix}
1 & 5 & -1 \\
3 & 7 & -11 \\
-2 & -2 & 10
\end{pmatrix}. \]

Is the vector
\[ \begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix} \]
in the column space of \( A \)? Justify your answer.

9. (a) If \( A \) is a \( 4 \times 3 \) matrix, what is the largest dimension of the row space of \( A \)?
(b) If \( A \) is a \( 3 \times 4 \) matrix, what is the largest dimension of the row space of \( A \)?

10. Let \( A \) be an \( m \times n \) matrix. Show that the null space of \( A \)
\[ \text{Nul} \, A = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \]
is closed under vector addition.

11. Suppose \( A \) is \( n \times n \) and for some \( \vec{b} \in \mathbb{R}^n \) the equation \( A\vec{x} = \vec{b} \) has more than one solution. Can the columns of \( A \) span \( \mathbb{R}^n \)? Why or why not? Explain.

12. True or False:
(a) If $A$ and $B$ are two $3 \times 3$ matrices and
\[ B = (\vec{b}_1, \vec{b}_2, \vec{b}_3) \]
then
\[ AB = (A\vec{b}_1 + A\vec{b}_2 + A\vec{b}_3). \]
(b) A plane in $\mathbb{R}^2$ is a two dimensional linear subspace of $\mathbb{R}^3$.
(c) If  
\[ \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \} \]
is a linearly independent set in a vector space $V$ then $\dim V \geq p$.
(d) If $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^3$ then the rank of the matrix $\vec{u}\vec{v}^T$ is always 0 or 1.
(e) For a $3 \times 3$ matrix $A$, $\det(3A) = 3 \det(A)$.

13. Let $f_0(t) = 1$, $f_1(t) = 1 + t$, $f_2(t) = 1 + t + t^2$, $f_3(t) = t^3$.
(a) Show that
\[ \mathcal{B} = \{ f_0(t), f_1(t), f_2(t), f_3(t) \} \]
is a basis for the vector space $P_3$ of all polynomials of degree at most 3.
(b) Find the coordinates of the polynomial $f(t) = t^2 + t^3$ relative to $\mathcal{B}$.

14. Let
\[ A = \begin{pmatrix} 2 & 1 & -1 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \]
Compute $\det A$. Find a basis for the column space of $A$. What is the rank and the nullity of $A$?

15. If
\[ f : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \]
is a linear map and $\text{Nul}(f) = \text{Span}\{\vec{e}_1\}$, what is the dimension of the image of $f$?

16. Is
\[ \mathcal{B} = \left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -6 \\ 11 \\ 10 \end{pmatrix} \right\} \]
a basis for $\mathbb{R}^3$?

17. Find $b$ such that $(-1, b, 2, 3)$ is in the span of $(1, 2, 3, 4)$ and $(3, 4, 4, 5)$.

18. Can you give a simple reason why $\det(A) = 0$?
(a) For which values of $\beta$ is $A$ invertible?

(b) Assuming $A$ is singular then find the rank of $A$, the nullity of $A$ and $\text{Nul}(A)$.

20. Suppose that $b_1$, $b_2$, $b_3$ and $b_4$ are real numbers. Show that there is exactly one polynomial $p(t)$ in the vector space $P_3$ of polynomials of degree at most 3 such that:

$$p(1) = b_1, \quad p'(0) = b_2, \quad \int_{-1}^{1} p(t) \, dt = b_3, \quad \text{and} \quad p(-1) = b_4.$$ 

21. Let 

$$H = \text{Span}\{\vec{u}, \vec{v}\} \quad \text{and} \quad K = \text{Span}\{\vec{u}, \vec{v}, \vec{u} + \vec{v}\}.$$ 

Prove that $H = K$.

22. Suppose that $A^2 = 0$. Prove that $A$ is not invertible.