Here are a slew of practice problems for the final culled from old exams:

1. Let $P_2$ be the vector space of polynomials of degree at most 2. Let

$$ B = \{ 1, (t - 2)^2, t^2 + t \}. $$

(a) Show that $B$ is a basis of $P_2$.
(b) Write $5t^2 + 5$ as a linear combination of the elements of $B$.

2. Let $A$ be the matrix:

$$
\begin{pmatrix}
2 & -1 & -1 \\
1 & 4 & 1 \\
-1 & -1 & 2
\end{pmatrix}
$$

(a) Determine the eigenvalues of $A$.
(b) Find a basis for each eigenspace of $A$.
(c) Diagonalise $A$.

3. Let $W$ be the subspace of $\mathbb{R}^4$ spanned by

$$
\vec{x}_1 = (1, -3, 0, 1) \quad \text{and} \quad \vec{x}_2 = (5, -5, -1, 2).
$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for $W$.
(b) Find the projection of

$$
\vec{y} = (1, 2, 1, 4)
$$

onto $W$.
(c) Find the distance from $\vec{y}$ to $W$.
(d) Find a basis for $W^\perp$.

4. Fully justify each answer.
(a) Show that the set of eigenvalues of a square matrix $A$ is the same as the eigenvalues of the matrix $A^T$.
(b) Suppose that $A$ is a square matrix with distinct eigenvalues $\lambda_1$ and $\lambda_2$. Suppose that $\vec{x}_1$ and $\vec{x}_2$ are (non-zero) eigenvectors with eigenvalues $\lambda_1$ and $\lambda_2$. Show that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.

5. Orthogonally diagonalise the following symmetric matrix

$$
\begin{pmatrix}
3 & -2 & 4 \\
-2 & 6 & 2 \\
4 & 2 & 3
\end{pmatrix}
$$
6. Find a least squares solution to $A\vec{x} = \vec{b}$ where:

$$A = \begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & 1 & -1
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
2 \\
5 \\
6
\end{pmatrix}.$$ 

Is this solution unique?

7. Suppose that $V$ is a vector space with subspaces $U$ and $W$. Justify your answer to the following by providing a proof or a counterexample:

(a) Is $U \cap W = \{ v \in V \mid v \in U \text{ and } v \in W \}$ a subspace of $V$?

(b) Is $U \cup W = \{ v \in V \mid v \in U \text{ or } v \in W \}$ a subspace of $V$?

8. As always, justify your answer.

(a) Is it possible that all solutions to a homogeneous system of 10 equations with 12 unknowns are multiples of all single non-zero vector?

(b) Is it possible for a system of 6 equations with 5 unknowns to have a unique solution for a fixed right hand side of constants.

(c) Show that if $A$ is diagonalisable and invertible, then so is $A^{-1}$.

9. (a) Let

$$\mathcal{D} = \{ f(x) \in P_n \mid f'(0) = 0 \}$$

denote the subset of the polynomials of degree at most $n$ whose derivative at zero is 0. Verify that $\mathcal{D}$ is a linear subspace of $P_n$.

(b) Suppose

$$A = \begin{pmatrix}
2 & -1 & 0 & 3 \\
-1 & 0 & 3 & 2
\end{pmatrix}.$$ 

Find a basis for $\text{Col}(A)$ and $\text{Nul}(A)$. What is rank$(A)$?

(c) Suppose that $A$ is an $m \times n$ matrix with $m < n$. Suppose rank$(A) < n$. Is it possible that the columns of $A$ span $\mathbb{R}^m$? Why or why not?

(d) Suppose that

$$H = \left\{ \begin{pmatrix} a + 2b + 3c \\ a + 2b + 3c \\ a + 2b + 3c \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}.$$ 

Find a basis for $H$.

10. (a) Suppose

$$A = \begin{pmatrix}
3 & -3 & 5 \\
0 & 4 & -3 \\
0 & 2 & -1
\end{pmatrix}.$$
Find the eigenvalues of $A$ with multiplicity.
(b) Diagonalise the matrix $A$ from (a).
(c) Suppose $B$ has eigenvalues 2 and 3 with corresponding eigenvectors $\vec{x}$ and $\vec{y}$ respectively. Suppose $\vec{z} = 10\vec{x} + 2\vec{y}$. Compute $B^{100}\vec{z}$. You may leave your answer in terms of $\vec{x}$ and $\vec{y}$.

11. (a) Use the Gram-Schmidt process to make

$$\mathcal{B} = \{ (1, -1), (2, 3) \}$$

into an orthogonal basis of $\mathbb{R}^2$.
(b) Find the least squares solution to $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 7 \end{pmatrix}.$$

12. For each statement, mark it true or false. If it is false give a (counter)example. If it is true give a reason - if the reason is a theorem, state the theorem, otherwise give a brief proof. No credit for answers without a correct reason or example. Unless explicitly stated, no assumptions are made on the dimensions of the matrices.
(a) If $A$ has $n$ different eigenvectors, then $A$ is diagonalisable.
(b) If $AP = PD$, where $D$ is diagonal, then the columns of $P$ are eigenvectors of $A$.
(c) If $\lambda$ is an eigenvalue of $A$ then $\lambda^{100}$ is an eigenvalue of $A^{100}$.
(d) An orthogonal matrix has orthonormal rows.
(e) If $AB$ is invertible and $A$ and $B$ are square then $A$ is invertible.
(f) If $\vec{x}_0$ is the least squares solution to $A\vec{x} = \vec{b}$ then $\vec{b}_0 = A\vec{x}_0$ is the closest vector in $\text{Col}(A)$ to $\vec{b}$.

13. (a) Suppose

$$A^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 2 & 3 & 3 \end{pmatrix}.$$ 

Solve $A\vec{x} = \vec{b}$.
(b) Define eigenvalue.
(c) Suppose that $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear function that reflects across the $x_1$-axis and then in the line $x_1 = x_2$. Find the matrix associated to $f$. 

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14. Find the general solution to
\[
\begin{align*}
   x_1 & + 5x_3 + 6x_4 = 6 \\
   x_1 + x_2 & + 2x_4 = 1 \\
   3x_1 + 2x_2 + 6x_3 + 11x_4 &= 11.
\end{align*}
\]

15. A linear function \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) is defined by
\[
f(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_3, x_2 - x_3, x_1 + 3x_2, x_3 + x_4).
\]
(a) Find the standard matrix for \( f \).
(b) Find a basis for the nullspace of \( f \).

16. For
\[
A = \begin{pmatrix}
2 & 2 & 4 & 5 & 0 \\
0 & 2 & 2 & 1 & 3 \\
1 & 1 & 3 & 4 & 2 \\
5 & 5 & 11 & 14 & 2
\end{pmatrix}
\]
find the following.
(a) rank of \( A \).
(b) a basis for the row space of \( A \).
(c) a basis for all \( \vec{b} \in \mathbb{R}^4 \) for which \( A\vec{x} = \vec{b} \) has a solution.

17. If the eigenvalues of \( A \) are 1, 2 and 3 what are the eigenvalues of \( A^{-1} \)? Give a brief reason for your answer.

18. For each statement, mark it True or False. If true, give a brief reason. If false, explain or give a counterexample. No credit if reason is wrong.
(a) If \( A \) and \( B \) are two \( 2 \times 2 \) matrices, with \( A \) invertible and if \( AB = 0 \), then \( B = 0 \).
(b) If \( A \) is a \( 2 \times 2 \) matrix which is diagonalisable, then \( A \) is symmetric.
(c) If the eigenvalues of a \( 3 \times 3 \) matrix are 0, 1 and 2, then \( A \) is diagonalisable.
(d) Suppose that \( A \) and \( B \) are two square matrices, and \( B \) is obtained from \( A \) by row operations. Then every eigenvalue of \( A \) is an eigenvalue of \( B \).

19. Let \( V \) be the plane in \( \mathbb{R}^4 \) spanned by the vectors \((2, 0, 1, 1)\) and \((1, 1, 0, 2)\). Find the vector in \( V \) closest to \( \vec{y} = (3, 1, 5, 1) \).

20. Determine if the set of vectors in \( \mathbb{R}^4 \),
\[
\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \}
\]
is linearly independent.
21. Suppose that you know the determinant of the matrix
\[
A = \begin{pmatrix}
1 & a & 2 \\
3 & b & 5 \\
-1 & c & -3
\end{pmatrix}
\]
is 3 and \( \vec{x} = (x_1, x_2, x_3) \) is a vector for which
\[
A \vec{x} = \begin{pmatrix}
2 \\
-1 \\
1
\end{pmatrix}.
\]
Find \( x_2 \).

22. True or false? (+1 pt for correct answer, -1 pt for incorrect answer).
(a) If \( A \) is any matrix the system \( A \vec{x} = \vec{0} \) must have at least one solution.
(b) If a square matrix is diagonalisable then its rows must be linearly independent.
(c) If \( V \) is a vector space and there is no set of \( n \) vectors which spans \( V \) then \( \dim(V) > n \).
(d) If there is a linearly dependent set
\[
\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \}
\]
of vectors in \( V \), then \( \dim(V) < 4 \).
(e) the function \( f(x_1, x_2) = (x_1, x_2 + 1) \) is a linear function from \( \mathbb{R}^2 \) onto \( \mathbb{R}^2 \).
(f) If \( A \) and \( B \) are two square matrices which are similar to each other, then they must have the same eigenvalues.
(g) If \( A \) and \( B \) are two square matrices which are similar to each other, then they must have the same eigenvectors.
(h) If \( A \) is a square matrix and 0 is an eigenvalue of \( A - 2I \) then 2 is an eigenvalue of \( A \).
(i) There is a linear function from \( \mathbb{R}^3 \) onto \( \mathbb{R}^4 \).

23. Give examples of \( 2 \times 2 \) matrices \( A \) and \( B \) with the same characteristic polynomial but \( A \) is diagonalisable and \( B \) is not.

24. In this problem \( A \) is a square matrix. Give a very brief answer for each question.
(a) If \( A \) is not invertible, what number must be an eigenvalue of \( A \)?
(b) If \( \dim \text{Nul}(A) = 1 \), what is the rank of \( A \)?
(c) If \( A \) is invertible what is the row reduced echelon form of \( A \)?
(d) If \( A \) is not invertible, find \( \det A \).
(e) If \( A^2 = 0 \), show that \( A \) is not invertible.

25. If
\[
\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}
\]
is a linearly independent set of vectors in a vector space and \( \vec{v}_4 \) is a vector in \( V \) which is not in the span of
\[
\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \},
\]
show (carefully) that
\[
\{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \},
\]
is a linearly independent set.

26. Find all eigenvalues and eigenvectors of
\[
\begin{pmatrix}
9 & -2 \\
2 & 5
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
6 & -2 & 0 \\
-2 & 9 & 0 \\
5 & 8 & 3
\end{pmatrix}.
\]

27. The matrix
\[
A = \begin{pmatrix}
2 & -1 & -1 \\
1 & 4 & 1 \\
-1 & -1 & 2
\end{pmatrix}
\]
has one eigenvalue equal to \(-2\). Diagonalise \( A \).

28. Let
\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]
Find a \( 3 \times 3 \) diagonal matrix \( D \) and a \( 3 \times 3 \) matrix orthogonal matrix \( U \) such that \( A = UDU^{-1} \). Compute \( A^{10} \).

29. Let \( \vec{u} = (-1, 0, 1, 1) \) and \( \vec{v} = (1, -1, -2, 0) \). Compute the length of \( \vec{u} \) and \( \vec{v} \).

30. Let \( \vec{u}_1 = (-1, 3, 1, 1) \), \( \vec{u}_2 = (6, -8, -2, -4) \) and \( \vec{u}_3 = (6, 3, 6, -3) \).
Let \( W = \text{Span}\{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \)
and let \( \vec{y} = (1, 0, 0, 1) \).
(a) Find an orthogonal basis of \( W \).
(b) Find the orthogonal projection \( \mathbb{P}_W(\vec{y}) \).
(c) Find the distance from \( \vec{y} \) to \( W \).
(d) Decompose the vector \( \vec{y} \) as follows: \( \vec{y} = \vec{y}_0 + \vec{y}_1 \), where \( \vec{y}_0 \in W \) and \( \vec{y}_1 \) is orthogonal to \( W \).

31. Let
\[
A = \begin{pmatrix}
1 & -1 \\
1 & 2 \\
-1 & 1
\end{pmatrix}
\quad \text{and} \quad
\vec{b} = \begin{pmatrix}
4 \\
0 \\
-2
\end{pmatrix}.
\]
(a) Find the orthogonal projection of \( \vec{b} \) onto \( \text{Col}(A) \).
(b) Find the least squares solution \( \vec{x}_0 \) such that
\[
\| \vec{b} - A\vec{x}_0 \| \leq \| \vec{b} - A\vec{x} \| \quad \text{for all} \quad \vec{x} \in \mathbb{R}^3.
\]
32. Let 
\[ \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}, \]
be three non-zero pairwise orthogonal vectors in \( \mathbb{R}^4 \). Prove that 
\[ \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \]
is a linearly independent set.

33. Let \( W \) be the subspace spanned by \( \vec{u}_1 = (1, -4, 0, 1) \) and \( \vec{u}_2 = (7, -7, -4, 1) \). Find an orthogonal basis for \( W \) by performing the Gram-Schmidt process to these vectors. Find a basis for \( W^\perp \).

34. True or false:
(a) If the matrices \( A \) and \( B \) are similar, that is, \( A = PBP^{-1} \) for some invertible matrix \( P \), then \( A \) and \( B \) have the same set of eigenvalues.
(b) Let \( \vec{v}_1, \vec{v}_2 \) and \( \vec{v}_3 \) be three vectors in \( \mathbb{R}^3 \). Then they are linearly independent if and only if they are pairwise orthogonal.
(c) If \( \det A = 0 \) then 0 is an eigenvalue of \( A \).
(d) The nullspace of an \( m \times n \) matrix \( A \) consists of all vectors in \( \mathbb{R}^n \) that are orthogonal to any vectors in the column space of \( A \).
(e) Let \( B \) be a \( 6 \times 8 \) matrix with \( \dim \text{Nul}(B) = 3 \). Then \( \text{rank}(B) = 3 \).
(f) \( A \) is diagonalisable if \( A = PDP^{-1} \) for some matrix \( D \) and some invertible matrix \( P \).
(g) Any invertible matrix is diagonalisable.
(h) Any upper triangular square matrix is diagonalisable.
(i) The inverse of a diagonalisable matrix is diagonalisable.
(j) An orthogonal matrix is orthogonally diagonalisable.
(k) Any three different eigenvectors of a matrix \( A \) corresponding to three different eigenvalues must be linearly independent.
(l) If \( \lambda \) is an eigenvalue of \( A \) then \( \lambda^2 \) is an eigenvalue of \( A^2 \).
(m) If the columns of \( A \) are linearly independent, the equation \( A\vec{x} = \vec{b} \) has exactly one least squares solution.
(n) If a square matrix has orthonormal columns then it has orthonormal rows.
(o) If a set 
\[ S = \{ \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n \} \]
has the property that \( \vec{u}_i \cdot \vec{u}_j = 0 \) whenever \( i \neq j \), then \( S \) is an orthonormal set.
(p) If \( A \) is a symmetric matrix and \( A\vec{x} = 2\vec{x}, A\vec{y} = 3\vec{y} \) then \( \vec{x} \cdot \vec{y} = 0 \).

35. True or false:
(a) The row space of \( A \) is the column space of \( A^T \).
(b) The inverse of an invertible \( n \times n \) matrix \( A \) can be found by row reducing the augmented matrix \( [A|I_n] \).
(c) If \( \mathbf{B} \) and \( \mathbf{C} \) are two bases of a vector space \( V \) then \( \mathbf{B} \) and \( \mathbf{C} \) have the same number of elements.
(d) Two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in \( \mathbb{R}^n \) are orthogonal if and only if their dot product is greater than or equal to \( 0 \).
(e) If \( \mathbf{B} \) is a basis of a vector space \( V \) and \( \mathbf{C} \) is a basis for a linear subspace then each vector in \( \mathbf{C} \) can be written as a linear combination of the vectors in \( \mathbf{B} \).
(f) The area of the parallelogram with vertices \((0, 0), (1, 0), (2, 3)\) and \((3, 3)\) is 9.
(g) If \( m < n \) then the columns of an \( m \times n \) matrix \( A \) could span \( \mathbb{R}^n \).
(h) If \( A \) and \( B \) are row equivalent then \( \text{Row}(A) = \text{Row}(B) \).
(i) If the second column of a matrix is a pivot column then \( x_2 \) is not a free variable.
(j) The determinant of a matrix is equal to the determinant of its transpose.
(k) A matrix with \( n \) distinct eigenvalues is diagonalisable.
(l) If \( A \) and \( B \) are row equivalent matrices then \( A \) and \( B \) have the same column space.
(m) A linearly independent set of vectors in \( \mathbb{R}^n \) containing \( n \) vectors is a basis of \( \mathbb{R}^n \).
(n) A set of more than \( n \) vectors in \( \mathbb{R}^n \) that spans \( \mathbb{R}^n \) is a basis of \( \mathbb{R}^n \).
(o) A set of vectors in \( \mathbb{R}^n \) that spans \( \mathbb{R}^n \) contains a basis of \( \mathbb{R}^n \).
(p) If \( V \) is a \( k \)-dimensional vector space and \( S \) is a set of \( k + 1 \) vectors in \( V \) then \( S \) contains a basis of \( V \).
(q) The Gram-Schmidt algorithm returns an orthogonal basis for a given subspace.
(r) If \( \det(A) = d \) then \( \det(kA) = k^n d \).
(s) If \( A \) is a \( 4 \times 4 \) matrix and the null space of \( A \) is a plane in \( \mathbb{R}^4 \) then the column space of \( A \) has a basis with two elements.
(t) If \( \mathbf{v} \) is a non-zero vector in \( V \) then
\[
\frac{\mathbf{v}}{\|\mathbf{v}\|}
\]
is a unit vector in the direction of \( \mathbf{v} \).

36. Let
\[
A = \begin{pmatrix}
5 & -1 & 1/2 \\
4 & 1 & 1 \\
0 & 0 & 3
\end{pmatrix}.
\]
(a) Calculate \( \det(A) \).
(b) Find \( A^{-1} \).
(c) Determine the span of the columns of \( A \).
(d) Find the characteristic polynomial of \( A \).
(e) Find the eigenvalue(s) of $A$.
(f) For each eigenvalue $\lambda$ you found in part (e) find a basis for the associated eigenspace $E_\lambda$.
(g) If possible, find a matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$. If not possible, explain why not.

37. Let

$$B = \begin{pmatrix}
1 & 2 & -3 & -4 & 5 & 6 \\
0 & 3 & -4 & 1 & 9 & 10 \\
0 & 2 & -1 & 0 & 8 & -2 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 4 & 0 & 0 & 8 & -3 \\
0 & 3 & 0 & 0 & -4 & 2
\end{pmatrix}$$

(a) Calculate the determinant of $B$ by choosing clever rows and columns along which to expand.
(b) What is the dimension of the row space of $B$? Justify your answer.
(c) What is the dimension of the column space of $B$? Justify your answer.

38. Please prove two of the following. Indicate clearly which two you would like graded.
(a) Assuming that $A$ is an invertible $n \times n$ matrix, show that the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution without appealing to the Invertible Matrix Theorem. Then show that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.
(b) If $V$ is a vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in $V$ prove that

$$\text{Span}\{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \}$$

is a linear subspace of $V$.
(c) Let $A$ be an $m \times n$ matrix. Prove that every vector in $\text{Nul}(A)$ is in the orthogonal complement of $\text{Row}(A)$. 