6. Quotient Groups II

Now suppose that we drop the hypothesis that $H$ is the kernel of a homomorphism and replace this with the hypothesis that $H$ is normal. Then we still get a group. The key thing to check is:

**Theorem 6.1.** Let $H$ be a subgroup of a group $G$. Then the following rule for multiplication of left cosets

$$(aH)(bH) = abH$$

if and only if $H$ is normal in $G$.

**Proof.** Suppose that this rule of multiplication is well-defined. Pick $a \in G$. We have to show that

$$aH = Ha.$$ 

Pick $x \in aH$. There are two possible ways to define the product $aHa^{-1}H$.

The really obvious way is to say

$$aHa^{-1}H = aa^{-1}H = eH = H.$$ 

As $aH = xH$ another way is to say

$$aHa^{-1}H = xHa^{-1}H = xa^{-1}H.$$ 

If the rule of multiplication is well-defined we must have

$$xa^{-1}H = H.$$ 

In this case $xa^{-1} \in H$, so that there is an element $h \in H$ such that $xa^{-1} = h$. Multiplying both sides of the right by $a$ we get

$$x = ha$$

so that $x \in Ha$. Thus the LHS is a subset of the RHS. By symmetry the RHS is a subset of the LHS. But then $aH = Ha$. As $a$ is arbitrary, $H$ is normal.

We have already proved that if $H$ is normal then multiplication of left cosets is well-defined but let’s prove it again.

Suppose that $H$ is normal. If $a'H = aH$ and $b'H = bH$ then $a' = ah_1$ and $b' = bh_2$, for some $h_1$ and $h_2$. On the other hand, we may find $h_3 \in H$ so that

$$h_1b = bh_3$$ since $bH = Hb$. 


It follows that
\[
a'b' = (ah_1)(bh_2)
= a(h_1b)h_2
= a(bh_3)h_2
= (ab)(h_3h_2).
\]

Thus
\[
a'b'H = abH,
\]
and the multiplication is indeed well-defined. \(\square\)

**Corollary 6.2.** Let \(H\) be a normal subgroup of a group \(G\).
Then the left cosets of \(H\) in \(G\) form a group with multiplication defined by
\[
(aH)(bH) = abH.
\]

**Proof.** We have already checked that this rule of multiplication is well-defined in (6.1). We first check associativity. Suppose that \(aH\), \(bH\) and \(cH\) are three left cosets. Then
\[
aH(bHcH) = aH(bcH)
= a(bc)H
= (ab)cH
= (abH)cH
= (aHbH)cH.
\]

Thus we have associativity.

We claim that \(eH = H\) is the identity. Suppose that \(aH\) is another left coset. We have
\[
aHeH = aeH = aH = eaH = eHaH.
\]

Thus \(eH\) is the identity.

Finally we check that \(a^{-1}H\) is the inverse of \(aH\). We have
\[
aHa^{-1}H = aa^{-1}H = eH = a^{-1}aH = a^{-1}HaH.
\]
Thus \(a^{-1}H\) is the inverse of \(aH\). \(\square\)

**Definition 6.3.** The group of left cosets with multiplication defined in (6.1) is called the **quotient group**, denoted \(G/H\).

**Example 6.4.** If \(n\mathbb{Z} = \langle n \rangle\) is the subgroup of \(\mathbb{Z}\) of all multiples of \(n\) then \(\langle n \rangle\) is a normal subgroup.
Indeed it is the kernel of the map
\[ \gamma : \mathbb{Z} \rightarrow \mathbb{Z}_n, \]
which sends a number to its remainder. The quotient group
\[ \mathbb{Z}/\langle n \rangle \]
is isomorphic to \( \mathbb{Z}_n \) the integers modulo \( n \).

**Example 6.5.** Let \( A_3 \subset S_3 \) be the alternating subgroup.

Then \( A_3 \) is the kernel of the homomorphism \( \phi : S_3 \rightarrow \mathbb{Z}_2 \) and so \( A_3 \) is normal. Let’s consider the group table of \( G = S_3 \).

\[
\begin{array}{c|cccccc}
* & e & (1, 2, 3) & (1, 3, 2) & (2, 3) & (1, 3) & (1, 2) \\
\hline
e & e & (1, 2, 3) & (1, 3, 2) & (2, 3) & (1, 3) & (1, 2) \\
(1, 2, 3) & (1, 2, 3) & (1, 3, 2) & e & (1, 2) & (2, 3) & (1, 3) \\
(1, 3, 2) & (1, 3, 2) & e & (1, 2, 3) & (1, 3) & (1, 2) & (2, 3) \\
(2, 3) & (2, 3) & (1, 3) & (1, 2) & e & (1, 2, 3) & (1, 3, 2) \\
(1, 3) & (1, 3) & (1, 2) & (2, 3) & (1, 3, 2) & e & (1, 2, 3) \\
(1, 2) & (1, 2) & (2, 3) & (1, 3) & (1, 2, 3) & (1, 3, 2) & e \\
\end{array}
\]

Note that things have been arranged so that the first three entries in each row (and so column) are the elements of \( A_3 \). The index of \( A_3 \) in \( S_3 \) is 2 = 6/3. So there are two left cosets, the elements of \( A_3 \) and everything else. If you shade the elements of \( A_3 \) one colour, say red, and everything else another colour, say blue, then we get the quotient group:

\[
\begin{array}{c|cc}
* & R & B \\
\hline
R & R & B \\
B & B & R \\
\end{array}
\]

Obviously this group is isomorphic to \( \mathbb{Z}_2 \).

On the other hand, the same thing won’t work if we start with the subgroup \( H = \{e, (1, 2)\} \). In this case there are three left cosets,

\[
H = \{e, (1, 2)\}, \quad (1, 2, 3)H = \{(1, 2, 3), (1, 3)\} \quad \text{and} \quad (1, 3, 2)H = \{(1, 3, 2), (2, 3)\}.
\]

Imagine these three cosets are represented by three colours, red, blue and yellow. How should we define the product of a red times a blue? Well, take a red element and a blue element and multiply them together. We could take \( e \) for a red and \((1, 2, 3)\) for a blue. Then \( e(1, 2, 3) = (1, 2, 3) \), which is blue. So, from this point of view, we should say red times blue is blue. But now suppose we take \((1, 2)\) and \((1, 3)\) another red and blue. Then \((1, 2)(1, 3) = (1, 3, 2)\), a yellow. So, from this point of view, we should say red times blue is yellow. So there is no consistent way to multiply red times blue.