HOMEWORK 4, DUE TUESDAY FEBRUARY 2ND

1. Let $R$ be a ring and let $I$ be an ideal of $R$, not equal to the whole of $R$. Suppose that every element not in $I$ is a unit. Prove that $I$ is the unique maximal ideal in $R$.

2. Let $\phi: R \rightarrow S$ be a ring homomorphism and suppose that $J$ is a prime ideal of $S$.
   (i) Prove that $I = \phi^{-1}(J)$ is a prime ideal of $R$.
   (ii) Give an example of an ideal $J$ that is maximal such that $I$ is not maximal.

3. Let $R$ be an integral domain and let $a$ and $b$ be two elements of $R$. Prove that
   (a) Show that $a | b$ if and only if $\langle b \rangle \subset \langle a \rangle$.
   (b) $a$ and $b$ are associates if and only if $\langle a \rangle = \langle b \rangle$.
   (c) Show that $a$ is a unit if and only if $\langle a \rangle = R$.

4. Prove that every prime element of an integral domain is irreducible.

5. (a) Show that the elements 2, 3 and $1 \pm \sqrt{-5}$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.
   (b) Show that every element of $R$ can be factored into irreducibles.
   (c) Show that $R$ is not a UFD.

Bonus Problems

6. Let $S$ be a commutative monoid, that is, a set together with a binary operation that is associative, commutative, and for which there is an identity, but not necessarily inverses. Treating this operation like multiplication in a ring, define what it means for $S$ to have unique factorisation.

7. Let $v_1, v_2, \ldots, v_n$ be a sequence of elements of $\mathbb{Z}^2$. Let $S$ be the semigroup that consists of all linear combinations of $v_1, v_2, \ldots, v_n$, with positive integral coefficients. Let the binary rule be ordinary addition. Determine which semigroups have unique factorisation.

8. Show that there is a ring $R$, such that every element of the ring is a product of irreducibles, whilst at the same time the factorisation algorithm can fail.