5. Field of fractions

The rational numbers \( \mathbb{Q} \) are constructed from the integers \( \mathbb{Z} \) by adding inverses. In fact a rational number is of the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers. Note that a rational number does not have a unique representative in this way. In fact

\[
\frac{a}{b} = \frac{ka}{kb}.
\]

So really a rational number is an equivalence class of pairs \([a, b]\), where two such pairs \([a, b]\) and \([c, d]\) are equivalent if and only if \( ad = bc \).

Now given an arbitrary integral domain \( R \), we can perform the same operation.

**Definition-Lemma 5.1.** Let \( R \) be any integral domain. Let \( N \) be the subset of \( R \times R \) such that the second coordinate is non-zero.

Define an equivalence relation \( \sim \) on \( N \) as follows.

\((a, b) \sim (c, d) \) if and only if \( ad = bc \).

**Proof.** We have to check three things, reflexivity, symmetry and transitivity.

Suppose that \((a, b) \in N\). Then

\[
a \cdot b = a \cdot b
\]

so that \((a, b) \sim (a, b)\). Hence \( \sim \) is reflexive.

Now suppose that \((a, b), (c, d) \in N\) and that \((a, b) \sim (c, d)\). Then \( ad = bc \). But then \( cb = da \), as \( R \) is commutative and so \((c, d) \sim (a, b)\). Hence \( \sim \) is symmetric.

Finally suppose that \((a, b), (c, d) \in R\) and that \((a, b) \sim (c, d), (c, d) \sim (e, f)\). Then \( ad = bc \) and \( cf = de \). Then

\[
(af)d = (ad)f
= (bc)f
= b(cf)
= (bc)d.
\]

As \((c, d) \in N\), we have \( d \neq 0 \). Cancelling \( d \), we get \( af = be \). Thus \((a, b) \sim (e, f)\). Hence \( \sim \) is transitive.

**Definition-Lemma 5.2.** The field of fractions of \( R \), denoted \( F \) is the set of equivalence classes, under the equivalence relation defined above. Given two elements \([a, b]\) and \([c, d]\) define

\[
[a, b] + [c, d] = [ad + bc, bd] \quad \text{and} \quad [a, b] \cdot [c, d] = [ab, cd].
\]
With these rules of addition and multiplication $F$ becomes a field. Moreover there is a natural injective ring homomorphism

$$\phi: R \longrightarrow F,$$

so that we may identify $R$ as a subring of $F$. In fact $\phi$ is universal amongs all such injective ring homomorphisms whose targets are fields.

Proof. First we have to check that this rule of addition and multiplication is well-defined. Suppose that $[a, b] = [a', b']$ and $[c, d] = [c', d']$. By commutativity and an obvious induction (involving at most two steps, the only real advantage of which is to simplify the notation) we may assume $c = c'$ and $d = d'$. As $[a, b] = [a', b']$ we have $ab' = a'b$. Thus

$$\begin{align*}
(a'd + b'c)(bd) &= a'bd^2 + bb'cd \\
&= ab'd^2 + bb'cd \\
&= (ad + bc)(b'd).
\end{align*}$$

Thus $[a'd + b'c, b'd] = [ad + bc, bd]$. Thus the given rule of addition is well-defined. It can be shown similarly (and in fact more easily) that the given rule for multiplication is also well-defined.

We leave it is an exercise for the reader to check that $F$ is a ring under addition and that multiplication is associative. For example, note that $[0, 1]$ plays the role of 0 and $[1, 1]$ plays the role of 1.

Given an element $[a, b]$ in $F$, where $a \neq 0$, then it is easy to see that $[b, a]$ is the inverse of $[a, b]$. It follows that $F$ is a field.

Define a map

$$\phi: R \longrightarrow F,$$

by the rule

$$\phi(a) = [a, 1].$$

Again it is easy to check that $\phi$ is indeed an injective ring homomorphism and that it satisfies the given universal property. \[\square\]

**Example 5.3.** If we take $R = \mathbb{Z}$, then of course the field of fractions is isomorphic to $\mathbb{Q}$. If $R$ is the ring of Gaussian integers, then $F$ is a copy of $a + bi$ where now $a$ and $b$ are elements of $\mathbb{Q}$.

If $R = K[x]$, where $K$ is a field, then the field of fractions is denote $K(x)$. It consists of all rational functions, that is all quotients

$$\frac{f(x)}{g(x)},$$

where $f$ and $g$ are polynomials with coefficients in $R$. 

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