Question 1. [10]

(a) Determine the left and right Riemann sums for the function \( f(x) = 1/(x + 1) \) on the interval \([0, n]\) with a partition into \( n \) intervals, each of length one.

(b) Find the exact area under the curve \( y = 1/(x + 1) \) above the \( x \)-axis for \( 0 \leq x \leq n \).

(c) Let \( H_n \) be the sum of reciprocals of the first \( n \) positive integers:

\[
H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.
\]

Prove that \( \ln(n+1) \leq H_n \leq \ln(n+1) + 1 \) for any positive integer \( n \).
Solutions. (a) The partition consists of the intervals $[0, 1], [1, 2], \ldots, [n - 1, n]$ each of length one. Consider the left and right endpoints of these intervals, the left Riemann sum is exactly $H_n$ whereas the right Riemann sum is

$$R_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n + 1} = \sum_{i=1}^{n} \frac{1}{i + 1}.$$  

(b) The exact area is the integral

$$\int_0^n \frac{1}{x + 1} \, dx = \ln(n + 1).$$

(c) The left Riemann sum in (a) overestimates the integral in (b) since $1/(x + 1)$ is decreasing, whereas the right Riemann sum in (a) underestimates the integral in (b). Therefore

$$R_n \leq \ln(n + 1) \leq H_n.$$  

Now $R_n$ is found from $H_n$ by taking away 1 and adding $1/(n + 1)$: that is $R_n = H_n - 1 + 1/(n + 1)$. Therefore

$$H_n - 1 + 1/(n + 1) \leq \ln(n + 1)$$  
which implies  
$$H_n < \ln(n + 1) + 1.$$  
Therefore we have what we want, $\ln(n + 1) \leq H_n \leq \ln(n + 1) + 1$. 

Question 2.  

State the first and the second fundamental theorems of calculus. Then evaluate each of the following expressions

\[ (a) \ \frac{d}{dt} \int_{1}^{t} e^{x^2} \, dx \quad (b) \ \frac{d}{dt} \int_{\sqrt{t}}^{1} \frac{\sin x}{x} \, dx. \]
Solution. The first part is bookwork. For (a), we use the second fundamental theorem:

\[ \frac{d}{dt} \int_1^t e^{x^2} \, dx = e^{t^2}. \]

We also use the second fundamental theorem for (b), namely with the chain rule. First we recall

\[ \int_{\sqrt{t}}^1 \frac{\sin x}{x} \, dx = -\int_1^{\sqrt{t}} \frac{\sin x}{x} \, dx. \]

Therefore

\[ \frac{d}{dt} \int_{\sqrt{t}}^1 \frac{\sin x}{x} \, dx = -\frac{d}{dt} \int_1^{\sqrt{t}} \frac{\sin x}{x} \, dx \]

\[ = -\frac{\sin \sqrt{t}}{\sqrt{t}} \cdot \frac{d}{dt} \sqrt{t} \]

\[ = -\frac{\sin \sqrt{t}}{2t}. \]
Question 3. [10]

(a) State the formula for the area swept out by a simple curve $r = f(\theta)$ in polar co-ordinates when $\alpha \leq \theta \leq \beta$.

(b) Determine the area between the two spirals $r = \theta$ and $r = 2\theta$ for $0 \leq \theta \leq 2\pi$. These are plotted below for $0 \leq \theta \leq 2\pi$. 

[Image of two spirals] Two spirals
Solution. Part (a) is bookwork. For part (b), we find the area inside the outer spiral and subtract the area inside the inner spiral from it. The area inside the outer spiral is

$$\frac{1}{2} \int_{0}^{2\pi} (2\theta)^2 d\theta = 2 \int_{0}^{2\pi} \theta^2 d\theta = \frac{16\pi^3}{3}.$$ 

The area inside the inner spiral is

$$\frac{1}{2} \int_{0}^{2\pi} \theta^2 d\theta = \frac{4\pi^3}{3}.$$ 

So the area between the spirals is $16\pi^3/3 - 4\pi^3/3 = 4\pi^3$. 
Question 4.

Consider a thin rod lying on the $x$-axis from $x = 0$ to $x = \frac{1}{2}$ where the density of the rod at position $(x, 0)$ is

$$\delta(x) = \frac{1}{\sqrt{1-x^2}}.$$

Find the center of mass of this rod.
Solution. The center of mass is computed by the formula

\[ \frac{\int_{a}^{b} x\delta(x)dx}{\int_{a}^{b} \delta(x)dx} \]

where \( \delta(x) \) is the density of the rod at position \( x \). In our case, this is

\[ \frac{\int_{0}^{1/2} \frac{x}{\sqrt{1-x^2}}dx}{\int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}}dx}. \]

To do the top integral, make the substitution \( u = 1 - x^2 \). Then

\[ \int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2} \int \frac{1}{\sqrt{u}}du = -\sqrt{u} + c = -\sqrt{1-x^2} + c. \]

So the top integral is

\[ \int_{0}^{1/2} \frac{x}{\sqrt{1-x^2}}dx = 1 - \sqrt{3/4}. \]

For the bottom integral, use the trig substitution \( x = \sin w \). Then \( \sqrt{1-x^2} = \cos w \) and \( dx/dw = \cos w \) so

\[ \int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{\cos w}{\cos w}dw = \int 1dw = w + c = \arcsin x + c. \]

Putting in the limits of integration we get

\[ \int_{0}^{1/2} \frac{1}{\sqrt{1-x^2}}dx = \arcsin(1/2) - \arcsin(0) = \pi/6. \]

Therefore the center of mass is at position

\[ \frac{1 - \sqrt{3/4}}{\pi/6} = \frac{6 - \sqrt{27}}{\pi} \approx 0.2558... \]

on the \( x \)-axis.
Question 5.

Find the work done in vertically lifting a solid conical container of height \( h \) meters and radius \( r \) meter through a height of 1 meter. You may use the fact that the acceleration due to gravity is 9.8 m/s\(^2\) and you may assume that the density is 1\( kg \) per unit of volume.

Conical container
Solution. Let \( m(x) \) denote the cross-sectional mass at height \( x \) up the cone. Then the work done is

\[
W = 9.8 \cdot \int_0^h m(x) \, dx.
\]

Now

\[
m(x) = \text{density} \times \text{area} = 1 \times \text{area}
\]

where area denotes the area of the cross-section at height \( x \). The cross section at height \( x \) is clearly a circle, but we need to know its radius. This is done using similar triangles. If \( r(x) \) is the radius of the circle at height \( x \), then

\[
\frac{r(x)}{r} = \frac{x}{h} \quad \text{therefore} \quad r(x) = \frac{rx}{h}.
\]

So the area of the cross section at height \( x \) is \( \pi(rx/h)^2 \) which is also \( m(x) \). Therefore

\[
W = 9.8 \int_0^h \pi(rx/h)^2 \, dx = 9.8 \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx = 9.8\pi \frac{r^2 h}{3}.
\]
Question 6. [10]

(a) Solve the differential equation \( \frac{dy}{dt} = -\alpha y \), where \( \alpha > 0 \) and where \( y(0) = 1 \).

(b) The element Francium has a half-life of 1300 seconds. If a container initially contains 1g/l of Francium, determine the number of seconds which elapse (to the nearest integer) until the amount of Francium left is 0.1g/l.
Solution. (a) Divide both sides by $y$ to get

$$\frac{dy}{y} = -\alpha dt.$$ Integrating both sides we get

$$\ln |y| = -\alpha t + c.$$ Exponentiating both sides we get

$$|y| = Ce^{-\alpha t}$$

for some constant $C$. Since $y(0) = 1$ we have $C = 1$ and since $y(t) \geq 0$ for all $t$ we take the positive absolute value. So we get

$$y = e^{-\alpha t}.$$ For (b), recall the half life is the time it takes for the quantity $y(t)$ to halve. Since $y(0) = 1$, the half-life is the value of $t$ such that $y(t) = 1/2$. In other words, we solve

$$e^{-\alpha t} = \frac{1}{2}.$$ Taking ln on both sides and dividing by $-\alpha$, we get

$$t = \frac{\ln 2}{\alpha}$$

is the half-life. Since Francium has a half-life of 1300 seconds,

$$1300 = \frac{\ln 2}{\alpha}$$

and so $\alpha = (\ln 2)/1300$. This means

$$y(t) = e^{-t(\ln 2)/1300}.$$ The time $t$ for which $y(t) = 0.1$ is found from

$$0.1 = e^{-t(\ln 2)/1300}.$$ Taking ln on both sides,

$$t = (1300 \ln 10)/(\ln 2) \approx 4318.506$$

is the time in seconds it takes to get to 0.1g/l.