Practice Final Examination #2

Math 154 – Combinatorics and Graph Theory

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
Calendars are allowed
State clearly any theorems used without proof
Total 50 points
Question 1.

(a) Let \( d \in \mathbb{N} \). Prove that the generating function for the number of compositions of \( n \) where each part is a multiple of \( d \) is

\[
\Phi(x) = \frac{1 - x^d}{1 - 2x^d}.
\]

(b) Prove that there are \( 2^{n/d - 1} \) such compositions if \( n \) is a multiple of \( d \).
Question 1...
Question 2.

(a) Let $S$ be the set of all binary strings not containing 110. Write down all the strings of length 1, 2 and 3 in $S$.

(b) Choose the expression uniquely creates all binary strings in $S$
   (1) $\{0\}^*\{\{1\}^*\{0\}^*\}^*\{1\}^*$
   (2) $\{0\}^*\{\{1\}\{0\}\{0\}^*\}^*\{1\}^*$
   (3) $\{0\}^*\{\{1\}\{11\}\{0\}^*\{0\}^*\}^*\{1\}^*$

(c) Write down the generating function for the number of strings of length $n$ in $S$.

(d) If $a_n$ is the number of strings of length $n$ in $S$, find a recurrence equation for $a_n$.

(e) Find $\lim_{n \to \infty} a_n^{1/n}$.
Question 2...
Question 3.

(a) State and prove the handshaking lemma. [3]
(b) Prove that a tree on \( n \) vertices has \( n - 1 \) edges. [4]
(c) Show that every tree has at least two vertices of degree one. [2]
(d)* Let \( T \) and \( U \) be edge-disjoint binary trees on a set \( V \) of vertices. Prove that if \( G = T \cup U \) then \( \lambda(G) = \delta(G) = 2 \). Is \( \kappa(G) = 2 \)? State any theorems used without proof. [3]
Question 3...
Question 4.

(a) Define $\Gamma(X)$ when $X \subseteq A$ is a set of vertices of a bipartite graph $G = (A \cup B, E)$.

(b) State Hall’s Theorem.

(c)* Let $n \in \mathbb{N}$. Prove that in a bipartite graph $G = (A \cup B, E)$ with $\delta(G) \geq n$ and $|A| = |B| = 2n$, there is a perfect matching using Hall’s Theorem.
Question 4...
Question 5.

Find a maximum \textit{st}-flow and minimum \textit{st}-cut in the network shown below. Show all working.

Figure 1: A network
Question 5...