Notation: For a set $S$ of configurations with weight function $\omega$, $\Psi_S^\omega$ denotes the exponential generating function of $S$ with respect to $\omega$, and $\Delta_S^\omega$ denotes the Dirichlet series for $S$ with respect to weight function $\omega$.

4 Exponential generating functions

If we try to compute the generating function for the set $S$ of all permutations, where the weight of a permutation is its length, then we get

$$\Phi_S(x) = \sum_{n=0}^{\infty} n! x^n$$

and there is no nice closed form formula for $\Phi_S(x)$—moreover this considered as a power series over $\mathbb{C}$ this has zero radius of convergence. It makes sense to rescale the problem by defining for a set $S$ of configurations with weight function $\omega$: the exponential generating function for $S$ with respect to $\omega$ is

$$\Psi_S^\omega(X) = \sum_{\sigma \in S} \frac{1}{\omega(\sigma)!} X^{\omega(\sigma)} = \sum_{k=0}^{\infty} a_k \frac{k!}{k!} X^k$$

where $a_k$ is the number of configurations of weight $k$ in $S$. In particular we see that for the set of permutations by length we get $\Psi(X) = 1/(1 - X)$, and the exponential generating function for all sequences of distinct elements of $[n]$ is easily seen to be $(1 + X)^n$. It turns out that generating functions of the form $\sum_{n=0}^{\infty} \frac{a_n}{n!} X^n$ have a combinatorial interpretation in terms of counting labeled structures. Such generating functions are called exponential generating functions and are differentiated from the ordinary generating functions we have encountered so far.

4.1 The Exponential Formula

For exponential generating functions, we’re going to find an analog of the product lemma for ordinary generating functions called the exponential formula. This formula is a special case of the composition lemma. First some notation. Let $T$ be any set of configurations. Each configuration $\tau \in T$ is assumed to be labeled in some way with a finite set of positive integers, called the label set of $\tau$, and denoted $\ell(\tau)$. Define the exponential generating function

$$\Psi_T(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

where $a_n$ is the number of configurations in $T$ which have a label set of size $n$ (so we’re assuming this depends only on $n$, not on the label set). The weight of a configuration
is given by the size of its label set. A natural example comes from graph theory: if $T$ consists of all connected graphs, then each $\tau \in T$ is a connected graph whose vertices are labeled with a set of positive integers and whose weight is the number of vertices in the graph. It is helpful to keep this example in mind throughout this section. Now here is the crucial definition, which tells us what kinds of configurations we can deal with when using exponential generating functions. For a set $T$ of labeled configurations, define $T^\oplus$ to be the set of all finite subsets of $T$ whose label sets form a partition of $[n]$ for some $n$. So a typical element of $T^\oplus$ is a family of disjoint sets $t_1, t_2, \ldots, t_k \in T$ such that for some $n$, we have $\ell(t_1) \sqcup \ell(t_2) \sqcup \cdots = [n]$. To fix ideas, consider again the example of graphs: $T$ is the set of all connected graphs. Then $T^\oplus$ is the set of all finite sets of connected graphs whose union of label sets is $[n]$ for some $n$. Since every graph is built out of connected pieces, or components, $T^\oplus$ comprises all finite labeled graphs on vertex set $[n]$. The weight of a configuration $\{t_1, t_2, \ldots, t_k\}$ in $T^\oplus$ is the sum of the weights of each $t_i$ -- so this is just the total number of labels used by $t_1, t_2, \ldots, t_k$. In the case of graphs given above, this is just the number of vertices in the graph. We also define $kT$ to be the set of all $\{t_1, t_2, \ldots, t_k\} \subset T$ such that $\ell(t_1) \sqcup \ell(t_2) \sqcup \cdots \sqcup \ell(t_k) = [n]$ for some $n$. In particular,

$$T^\oplus = \bigcup_{k=1}^\infty kT$$

and these are the analogues of $T^*$ and $T^k$. We say that the elements of $kT$ or $T^\oplus$ are uniquely created if each element can be represented in exactly one way as a set labeled configurations in $T$.

**Theorem 1 (Exponential Formula)** Let $T$ be a set of configurations with a given weight function, and suppose that the elements of $kT$ and $T^\oplus$ are uniquely created and no configurations have weight zero. Then

$$\Psi_T^\oplus(X) = \exp(\Psi_T(X)) \quad \text{and} \quad \Psi_{kT}(X) = \frac{\Psi_T(X)^k}{k!}.$$ 

**Proof** The first statement follows immediately by the sum lemma and the second statement, and the fact that $\Psi_T(X)^k/k!$ is summable. Recalling the multinomial coefficients $\binom{m}{n_1 n_2 \ldots n_k}$ which represent the number of ways to partition a set of $m$ elements into sets of $n_1, n_2, \ldots, n_k$ elements, we observe

$$\Psi_{kT}(X) = \sum_{m=0}^\infty \frac{[\omega^{-1}(m)]}{m!} X^m$$

$$= \sum_{m=0}^\infty \frac{1}{m!} \binom{m}{n_1 n_2 \ldots n_k} \frac{1}{k!} \prod_{i=1}^k a_{n_i} X^{n_1 + n_2 + \cdots + n_k}$$

$$= \sum_{n_1, n_2, \ldots, n_k} \frac{1}{k! n!} \alpha_{n_i} X^{n_i}$$

$$= \frac{1}{k!} \left( \sum_{n=0}^\infty \frac{1}{n!} X^n \right)^k = \frac{1}{k!} \Psi_T(X)^k.$$
This completes the proof.

4.1.1 Stirling numbers of the first kind

Recall $s_{n,k}$ is the number of permutations of $[n]$ with $k$ cycles, where $n \geq 1$ and $k \geq 0$ and we define $s_{n,0} = 0$ for all $n \geq 1$. We determine the generating function for $S$, the set of all permutations with $k$ cycles, where the weight function is the length of the permutation. We know $S = kT$ where $T$ is the set of cycles, and since

$$\Psi_T(X) = \sum_{n=1}^{\infty} \frac{(n-1)!}{n!} X^n = -\log(1 - X),$$

the exponential formula gives

$$\Psi_S(X) = \frac{1}{k!} \Psi_T(X)^k = \frac{1}{k!} (-\log(1 - X))^k.$$ 

So it follows that

$$\sum_{n=1}^{\infty} \frac{s_{n,k}}{n!} X^n = \frac{1}{k!} (-\log(1 - X))^k.$$ 

Just as with compositions of $n$, we can ask for permutations of $[n]$ with restrictions on the cycles in the permutation. All this requires is modification of $\Psi_T(X)$ in the above computation. For instance, in the case all cycles are even,

$$\Psi_T(X) = \frac{1}{2} \log \frac{1}{1 - X^2}.$$ 

By the exponential formula,

$$\Psi_{T \circ}(x) = \exp(\Psi_T(x)) = (1 - X^2)^{-\frac{1}{2}}$$

is the exponential generating function for the permutations of $[n]$ with only even cycles. So the number of such permutations (when $n$ is even of course) is

$$n! \left(\frac{-1/2}{n/2}\right) (-1)^{n/2} = \frac{n!}{2^n} \binom{n}{n/2}.$$ 

One remark is that we can now determine the two-variable generating function for Stirling numbers of the first kind, namely

$$\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} s_{n,k} \frac{X^n}{n!} Y^k = \exp(-Y \log(1 - X)) = (1 - X)^{-Y}.$$ 

Often one finds instead the generating function for signed Stirling numbers, namely $(-1)^{n-k} s_{n,k}$ which is readily computed to be $(1 + X)^Y$. 
4.1.2 Bell numbers and Stirling numbers of the second kind

Perhaps the simplest combinatorial question is to determine the total number of partitions of \([n]\), which are called Bell numbers \(B(n)\). Recall the Stirling numbers (of the second kind) count the partitions with \(k\) parts, and are denoted \(S_{n,k}\) and

\[
B(n) = \sum_{k=0}^{n} S_{n,k}.
\]

The convention is \(S_{n,0} = 0\) for \(n \in \mathbb{N}\). Let \(T = \{[k] : k \in \mathbb{N}\}\) with weight function \(\omega(\tau) = |\tau|\). The label set for \([k]\) is a set of positive integers of size \(k\) and only \([k]\) has a label set of size \(k\). Then \(T^\oplus\) is the union of partitions of \([n]\) for \(n \in \mathbb{N}\), and

\[
\Psi_T(X) = \sum_{n \in \mathbb{N}} \frac{1}{n!} X^n = \exp(X) - 1.
\]

By the exponential formula,

\[
\Psi_{T^\oplus}(X) = \exp(\exp(X) - 1)
\]

is the strikingly neat generating function for Bell Numbers \(B(n)\). Similarly

\[
\Psi_{kT}(X) = \frac{(\exp(X) - 1)^k}{k!}
\]

is the generating function for the Stirling numbers \(S_{n,k}\) with \(k\) fixed. If we want the bivariate generating function for the numbers \(S_{n,k}\), namely

\[
\Psi(X, Y) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} S_{n,k} \frac{X^n}{n!} Y^k = \sum_{k=0}^{\infty} \frac{\Psi_{kT}(X)^k}{k!} Y^k = \exp(Y \exp(X) - Y) - \exp(X) + 1.
\]

Here we consider \(\Psi\) as a generating function over the ring \(\mathbb{C}[[X]]\).

4.2 Counting labeled trees

Cayley proved that there are \(n^{n-2}\) labeled trees where \(n \geq 2\). If a particular vertex – called the root is distinguished – then the answer is \(n^{n-1}\) for \(n \in \mathbb{N}\). We only consider non-empty trees. We can recover this using exponential generating functions. First note that if \(\Psi_T(X)\) is the exponential generating function for rooted labeled trees, where \(\Psi_T(0) = 0\), and \(\Psi_F(X)\) is the generating function for rooted labeled forests (each component of which is rooted), then

\[
\Psi_F(X) = \exp(\Psi_T(X)).
\]

From each forest, we can obtain a tree with one more vertex by adding a new root joined to the root in each component of the forest. Therefore

\[
\Psi_T(X) = X \Psi_F(X)
\]
and we obtain an implicit formula for the generating function, namely

\[ \Psi_T(X) = X \exp(\Psi_T(X)). \]

It is not clear at this stage how to obtain coefficients from this formula, but it leads us to the Lagrange Inversion Theorem.

5 Dirichlet series

In the last section we saw how to determine the exponential generating function for labeled structures which decompose uniquely into labeled substructures. We describe another generating function which is especially adapted to multiplicative problems in number theory. Let \( s \in \mathbb{N} \). If \( S \) is a set of configurations with weight function \( \omega \), then the Dirichlet series for \( S \) with respect to \( \omega \) is

\[ \Delta_S^\omega(X) = \sum_{\sigma \in S} \omega(\sigma)^{-X}. \]

If \( a_n \) is the number of configurations of weight \( n \) then this is

\[ \Delta_S^\omega(X) = \sum_{n=0}^{\infty} a_n n^{-X}. \]

A very special Dirichlet series called the Dirichlet \( L \)-series generates characters:

\[ L(X, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{nX} \]

where \( \chi \) is completely multiplicative character on \( \mathbb{Z}/m\mathbb{Z} \). When \( \chi \equiv 1 \) then we obtain the Riemann zeta function:

\[ \zeta(X) = \sum_{n=1}^{\infty} \frac{1}{nX}. \]

Here is the first fundamental combinatorial lemma for Dirichlet series.

**Lemma 2 (Product lemma for Dirichlet series)** Let \( A, B \) be sets of configurations and suppose that \( \alpha, \beta, \gamma \) are weight functions on \( A, B \) and \( A \times B \) respectively such that \( \gamma(\sigma, \tau) = \alpha(\sigma) \cdot \beta(\tau) \). Then

\[ \Delta_{A \times B}^\gamma(X) = \Delta_A^\alpha \cdot \Delta_B^\beta. \]

**Proof** This is straight from the definition of multiplication:

\[
\begin{align*}
\Delta_{A \times B}^\gamma(X) & = \sum_{\sigma \in A} \sum_{\tau \in B} (\alpha(\sigma) \beta(\tau))^{-X} \\
& = \sum_{\sigma \in A} \alpha(\sigma)^{-X} \cdot \sum_{\tau \in B} \beta(\tau)^{-X}. 
\end{align*}
\]
An immediate application is to find the Dirichlet series for ordered factorizations of positive integers, in other words, let $a_n$ denote the number of sequences $(n_1, n_2, \ldots)$ such that $n_1 \cdot n_2 \cdots = n$ with $n_i \in \mathbb{N}\{1\}$ for all $i$. With slight abuse of notation, let $S^*$ denote the set of all sequences with entries from a set $S$. Then with $S = \mathbb{N}\{1\}$,

$$a_n = \lfloor n^{-X} \Delta_{S^*}(X) \rfloor = \lfloor n^{-X} \frac{1}{1 - \Delta_S(X)} \rfloor,$$

by the product lemma. Now

$$\Delta_S(X) = \sum_{n \in S} \frac{1}{n^X} = \sum_{n=2}^{\infty} \frac{1}{n^X} = \zeta(X) - 1.$$

Therefore

$$a_n = \lfloor n^{-X} \Delta_S(X) \rfloor = \lfloor n^{-X} \frac{1}{1 - \zeta(X)} \rfloor.$$

If we wanted the number of factorizations with $k$ factors, we would have

$$a_{n,k} = \lfloor n^{-XY^k} \frac{1}{1 + Y - Y\zeta(X)} \rfloor.$$

We are going to return to this topic when we discuss number theoretic results and, in particular, normal orders.

### 6 Linear Recurrence Equations

A linear recurrence equation with constant coefficients for a sequence $(a_n)_{n \in \mathbb{N}}$ has order $k$ if it can be written as

$$b_k a_n + b_{k-1} a_{n-1} + \cdots + b_0 a_{n-k} = f(n)$$

for some constants $b_0, b_1, \ldots, b_k$ and some function $f$. The following theorem gives a method for solving such equations:

**Theorem 3** Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers satisfying $b_k a_n + b_{k-1} a_{n-1} + \cdots + b_0 a_{n-k} = f(n)$ for some constants $b_0, b_1, \ldots, b_k \in \mathbb{C}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_j$ be the distinct roots of the equation

$$b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0 = 0$$

such that $\alpha_i$ has algebraic multiplicity $m_i - 1$, and let $(c_n)_{n \in \mathbb{N}}$ be any solution to the recurrence equation. Then

$$a_n = P_1(n) \alpha_1^n + P_2(n) \alpha_2(n) + \cdots + P_j(n) \alpha_j^n + c_n$$

where each $P_i$ is a polynomial of degree at most $m_i - 1$ whose coefficient depend only on $a_1, a_2, \ldots, a_k$.

**Proof** Let $\Phi(X)$ denote the generating function for $(a_n)_{n \in \mathbb{N}}$ and let $\Phi_i(X) = a_0 + a_1 X + \cdots + a_i X^i$. First consider the case $f \equiv 0$, in which case we take $c_n = 0$ for all $n$. Then adding up the recurrence equation multiplied by $X^n$ for $n \geq k$ gives

$$\sum_{i=1}^{k} (\Phi(X) - \Phi_{k-i}(X)) b_{k-i+1} + b_0 X^k \Phi(X) = 0.$$
In other words, 

\[ \Phi(X) = \frac{\sum_{i=0}^{k-1} b_{k-i}X^i \Phi_{k-i-1}(X)}{\sum_{i=0}^{k} b_i X^{k-i}} = \frac{P(x)}{\prod_{i=1}^{j}(1 - \alpha_i X^{m_i})}. \]

From basic linear algebra, there exist unique constants \( c_{hi} : 1 \leq h \leq j, 1 \leq i \leq m_h \) such that

\[ \Phi(X) = \sum_{h=1}^{j} \sum_{i=1}^{m_h} c_{hi} \frac{(n + i - 1) \alpha_i^n}{(1 - \alpha_i X)^i}. \]

By the binomial theorem,

\[ a_n = [X^n] \Phi(X) = \sum_{h=1}^{j} \sum_{i=1}^{m_h} c_{hi} \binom{n + i - 1}{i - 1} \alpha_i^n = P_1(n) \alpha_1^n + P_2(n) \alpha_2(n) + \cdots + P_j(n) \alpha_j^n \]

where each \( P_i \) is a polynomial of degree at most \( m_i - 1 \) whose coefficient depend only on \( a_1, a_2, \ldots, a_k \). If \( f \neq 0 \), then note that \( a_n - c_n \) is a solution to the equation with \( f \equiv 0 \) and therefore from the first part \( a_n - c_n = P_1(n) \alpha_1^n + P_2(n) \alpha_2(n) + \cdots + P_j(n) \alpha_j^n \).

The proof of the theorem tells us how to switch between rational generating functions and linear recurrences with constant coefficients. Specifically, if \( \Phi(X) \) is the generating function for \( (a_n)_{n \in \mathbb{N}} \) and

\[ \Phi(X) = \frac{P(X)}{Q(X)} \]

for some polynomial \( Q(X) \), then \( (a_n)_{n \in \mathbb{N}} \) satisfies a homogeneous recurrence equation whose characteristic equation is

\[ x^{\deg(Q)} \cdot Q(x^{-1}) = 0. \]

In other words, the recurrence equation \( b_k a_n + b_{k-1} a_{n-1} + \cdots + b_0 a_{n-k} = 0 \) corresponds to the rational generating function

\[ \Phi(X) = \frac{P(X)}{b_0 X^k + b_1 X^{k-1} + b_{k-1} X + b_k} \]

where \( P(X) \) is a polynomial.

**Simple examples.**

For instance, suppose we want to solve \( F_n = F_{n-1} + F_{n-2} \) with \( F_1 = F_2 = 1 \). The characteristic equation is \( x^2 - x - 1 = 0 \) with distinct roots \( x = (1 \pm \sqrt{5})/2 \), for convenience we call these \( \varphi \) and \( -\varphi \) respectively. Therefore

\[ F_n = P_1 \varphi^n + P_2 \varphi^{-n} \]

for some constants \( P_1 \) and \( P_2 \). One sees quickly that \( P_1 = 1/\sqrt{5} \) and \( P_2 = -1/\sqrt{5} \) from \( a_1 = a_2 = 1 \). Now suppose we want to solve \( a_n = a_{n-1} + a_{n-2} + (-1)^n \) where \( a_1 = 0 \)
and $a_2 = 1$. We have to guess a particular solution to this nonhomogeneous equation (generally a very difficult task). In this case, clearly $(-1)^n$ is a solution so we have $a_n$ is $P_1 \phi^n + P_2 \overline{\phi}^n + (-1)^n$ where $P_1$ and $P_2$ are constants. The Lukas numbers are $L_n = \phi^n + \overline{\phi}^n$ and satisfy $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1$ and $L_2 = 3$. The solution is then

$$a_n = (-1)^n + 3F_n - L_n.$$  

Finally, if we consider the equation $a_n = 2a_{n-1} - a_{n-2} + n$ with $a_0 = a_1 = 1$, the characteristic equation has root $x = 1$ with multiplicity 2, so a linear function of $n$ solves the homogeneous equation. We check by trial and error that a cubic function of $n$ solves the non-homogeneous equation, from which we get $a_n = 1 + \frac{1}{6}(n - 1)n(n + 4)$.

### 6.1 General recurrence equations

One can use the algebra of rings of formal power series to some extent, for instance, to turn the recurrences into differential or integral equation. This also requires careful choice of which type of generating function to use. We have already seen that if $\Phi(X)$ is the generating function for $(a_n)_{n \in \mathbb{N}}$, then the following basic calculus rules hold:

**Rules of calculus on ordinary generating functions.**

1. $a_{n+k} \leftrightarrow X^{-k}(\Phi(X) - \sum_{i=0}^{k-1} a_i X^i)$. \textbf{[Shift]}
2. $n^k a_n \leftrightarrow (X \partial)^k \Phi$. \textbf{[Derivative]}
3. $a_n/(n+k) \leftrightarrow (\frac{1}{X} \int)^k \Phi$. \textbf{[Integral]}
4. $\sum_{n_1 + \cdots + n_k = n} \prod_{i=1}^{k} a_{n_i} \leftrightarrow \Phi^k$. \textbf{[Composition 1]}
5. $\sum_{k=0}^{\infty} \sum_{n_1 + \cdots + n_k = n} \prod_{i=1}^{k} a_{n_i} \leftrightarrow 1/(1 - \Phi)$. \textbf{[Composition 2]}
6. $\sum_{k=0}^{n} a_k \leftrightarrow \Phi/(1 - X)$. \textbf{[Partial Sums]}

For instance, suppose we want to solve for $n \geq 1$ the recurrence

$$a_n = \sum_{k=0}^{n} k(a_0 + a_1 + \cdots + a_k)$$

with $a_0 = 0$ and $a_1 = 1$. This is a recurrence with non-constant coefficients, and is equivalent to

$$\Phi(X) = X \partial \left( \frac{\Phi(X)}{1 - X} \right) \cdot \frac{1}{1 - X}.$$  

We could rewrite this as

$$X(1 - X) \partial \Phi = ((1 - X)^3 - X) \Phi$$

but then we are not permitted to divide by $X(1 - X)$. Instead let $\Upsilon = \Phi/X$ (which is well-defined) to obtain

$$\frac{\partial \Upsilon}{\Upsilon} = X - \frac{X}{(1 - X)^3} - \frac{X^2}{1 - X)^3}.$$  

A quick formal integration yields

\[ \Upsilon(X) = C(1 - X) \exp(-2X + \frac{1}{2}X^2) \]

Now this implies

\[ \Phi(X) = CX(1 - X) \exp(-2X + \frac{1}{2}X^2) \]

and since \( \partial \Phi(0) = 1 \) we get \( C = 1 \) and

\[ \Phi(X) = X(1 - X) \exp(-2X + \frac{1}{2}X^2) \]

We could explicitly determine \( a_n \) by expanding these quantities as formal power series: for \( n \geq 2 \),

\[ a_n = 2^{n-2} \sum_{k=0}^{n} \frac{(-2)^{-4k}}{k!(n - 3k - 2)!} \left( \frac{2}{n - 3k - 1} - 1 \right) \]

and the first few values are \( a_2 = 1, a_3 = 0, a_4 = a_5 = -\frac{7}{6}, a_6 = -\frac{2}{5} \).

However, especially for systems of recurrence equations, exponential generating functions \( \Psi(X) \) are better. Here are three basic rules:

**Rules of calculus on exponential generating functions.**

1. \( a_{n+k} \leftrightarrow \partial^k \Psi \). [Shift]
2. \( n^k a_n \leftrightarrow (X\partial)^k \Psi \). [Derivative]
3. \( \sum_{n_1 + \ldots + n_k = n} \frac{1}{n!} (n_{n_2 \ldots n_k}) \prod_{i=1}^{k} a_{n_i} \leftrightarrow \Psi^k \). [Composition]

For example the shift operation immediately reduces linear recurrences to differential equations. The reader is encouraged to reprove the theorem and try some recurrences using exponential generating functions, and also to summarise rules for Dirichlet series which can be used to solve recurrences. For systems of linear equations with constant coefficients, we may use these rules to convert the system to a system of linear differential equations with constant coefficients, and this can be solved using linear algebra. We remark for instance that the Fibonacci equation is \( A''(X) = A'(X) + A(X) \) with \( A(0) = 1 \) and \( A'(0) = 1 \). However, in the end, non-linear recurrences have to be solved via some ad-hoc techniques, which we are not going to address here.