Question 1

Let $X$ be the outcome on a fair 6 sided die and let $Y$ be the outcome on a fair 12 sided die. Determine

a. $E(X)$ and $E(Y)$.

b. $E(X + Y)$.

c. $E(XY)$ if $X$ and $Y$ are independent.

Solutions. (a)

\[ E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{7}{2} \]
\[ E(Y) = 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + \cdots + 12 \cdot \frac{1}{12} = \frac{13}{2} \]

(b) $E(X + Y) = E(X) + E(Y) = 10$ by linearity of expectation.

(c) $E(XY) = E(X)E(Y) = \frac{91}{4}$ by independence.
Question 2

A fair coin is tossed 9 times independently. Let \(X\) be the number of heads and \(Y\) the number of tails. Determine exactly the following quantities.

a. \(P(Y = 3)\).

b. \(P(X \geq 4)\).

c. \(E(2^X)\).

d. \(P(|X - Y| = 1)\).

**Solutions.** (a) We know \(X, Y \sim \text{Bin}(9, 1/2)\). Therefore

\[
P(Y = 3) = \binom{9}{3} \left(\frac{1}{2}\right)^3 \left(1 - \frac{1}{2}\right)^6 = \binom{9}{3} \left(\frac{1}{2}\right)^9 = \frac{21}{128}.
\]

(b) \(P(X \geq 4) = 2^{-9} \sum_{x \geq 4} \binom{9}{x} = 2^{-9} (126 + 126 + 84 + 36 + 9 + 1) = \frac{191}{256}\).

(c) Using the binomial theorem:

\[
E(2^X) = 2^{-9} \sum_{x=0}^{9} \binom{9}{x} 2^x = 2^{-9} (1 + 2)^9 = \left(\frac{3}{2}\right)^9.
\]

(d) Note that \(X + Y = 9\) and therefore if \(|X - Y| = 1\) then \(X = 4\) and \(Y = 5\) or \(X = 5\) and \(Y = 4\). So the answer is

\[
P(X = 4, Y = 5) + P(X = 5, Y = 4) = \binom{9}{4} 2^{-9} + \binom{9}{5} 2^{-9} = \frac{63}{128}.
\]
Question 3.

State and prove Chebyshev’s Inequality.
Question 4.

State the weak law of large numbers and the central limit theorem for independent random variables $X_1, X_2, \ldots$ with mean $\mu$ and variance $\sigma^2$. Show that the central limit theorem implies the weak law of large numbers.

Solution. The central limit theorem says

$$P\left(\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma \sqrt{n}}\right) \xrightarrow{d} Z$$

where $Z \sim N(0, 1)$. Let’s get the weak law of large numbers from this. Fix $\varepsilon > 0$ and let $Y_n = X_1 + X_2 + \cdots + X_n = n \bar{X}_n$. Then

$$P(|\bar{X}_n - \mu| > \varepsilon) = P(|Y_n - n\mu| \leq \varepsilon n)$$

$$= P\left(\left|\frac{Y_n - n\mu}{\sigma \sqrt{n}}\right| \leq \frac{\varepsilon \sqrt{n}}{\sigma}\right).$$

Let $K = \frac{\varepsilon \sqrt{n}}{\sigma}$, and note that $K \to \infty$. So by the central limit theorem, this probability is asymptotic to

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = 1.$$

So we have shown as $n \to \infty$ that

$$P(|\bar{X}_n - \mu| \leq \varepsilon) \to 1$$

which is the weak law of large numbers.