5 Graphs

Some of the Putnam problems are to do with graphs. They do not assume more than a basic familiarity with the definitions and terminology of graph theory.

5.1 Basic definitions of graph theory

A graph is a pair \((V, E)\) where \(V\) is a set and \(E\) is a set of unordered pairs of elements of \(V\). The elements of \(V\) are called vertices and the elements of \(E\) are called edges. The degree of a vertex \(v \in V\) is the number of edges containing \(v\), and denoted \(d(v)\). A ubiquitous lemma in graph theory states

\[ \sum_{v \in V} d(v) = 2|E| \]

in words, when we add up the degrees we get twice the number of edges. This is called the handshaking lemma, and it is a prime example of counting something in two ways. Namely, when we add up the degrees of the graph, each edge is counted twice, one for each of its endpoints, and that is why we get \(2|E|\).

If \(G = (V, E)\) is a graph and \(X\) is a set of vertices of \(G\), then \(G - X\) denotes the graph \((V \setminus X, F)\) where \(F\) is the set of edges of \(G\) contained in \(V \setminus X\). If \(L\) is a set of edges of \(G\), then \(G - L\) denotes the graph \((V, E \setminus L)\) – we just remove all edges in \(L\) from \(E\). A subgraph of a graph \((V, E)\) is a graph \((W, F)\) such that \(W \subseteq V\) and \(F \subseteq E\). A walk of length \(k\) in a graph is an alternating sequence \((v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1})\) of vertices and edges of the graph such that \(e_i = \{v_i, v_{i+1}\}\) for \(1 \leq i \leq k\). Note that some vertices and edges might be repeated since it is a sequence. A walk of length \(k\) is a path of length \(k\) if it has no repeated vertices (and therefore also no repeated edges). A walk of length \(k\) as above is closed if \(v_{k+1} = v_k\). If in addition none of the other vertices is repeated, then it is called a cycle of length \(k\). A graph \(G = (V, E)\) is bipartite if \(V\) has a partition \((A, B)\) with \(A, B \neq \emptyset\) such that every edges in \(E\) has one vertex in \(A\) and one vertex in \(B\). The complete graph \(K_n\) on \(n\) vertices is the graph with all possible edges on \(n\) vertices: there are \(\binom{n}{2}\) edges in this graph; every vertex is joined by an edge to every other vertex. For \(n = 3\), it is a triangle (cycle of length 3).
A graph is connected if between any two vertices of the graph there is a walk. A basic example of a connected graph is a tree: the definition of a tree states that a tree is graph which has no cycles and yet is connected. The connected graph on \( n \) vertices with the smallest number of edges (minimally connected graph) is a tree. Every tree on \( n \) vertices has \( n - 1 \) edges. A graph is planar if it can be drawn in the plane with the vertices as distinct points and the edges as smooth curves between their vertices in such a way that no two edges cross, meaning that if \( e \) and \( f \) are edges and \( p \) is a point in common to \( e \) and \( f \), then \( p \) is a vertex of \( e \) and a vertex of \( f \). The connected regions obtained by removing the drawing from \( \mathbb{R}^2 \) are called faces. The size of a face is the length of the closed walk completing its boundary. By the handshaking lemma, if we add up the face sizes of a graph drawn in the plane, we get twice the number of edges.

### 5.2 Standard Results

**Problem 1.** Prove that every graph on at least two vertices has two vertices of the same degree.

**Solution** The degrees of vertices in a graph on \( n \) vertices are integers in \{0, 1, 2, \ldots, n - 1\}. If one of them is zero, then every other vertex has degree at most \( n - 2 \), so we have that the degrees are all integers in \{0, 1, 2, \ldots, n - 2\}. By the pigeonhole principle, two of the degrees must be equal. If none of them is zero, then the degrees are numbers in \{1, 2, \ldots, n - 1\}, and again the pigeonhole principle shows two of them are equal. \(\square\)

**Problem 2.** Prove that the number of vertices of odd degree in a graph is even.

**Solution** This comes from the handshaking lemma. Let \( d(v) \) be the degree of a vertex \( v \) in a graph. Then we know

\[
\sum_{v \in V} d(v) = 2|E|
\]

where \( E \) is the set of edges in the graph. Now on the left side, there cannot be an odd number of odd \( d(v) \)'s, since then the sum on the left would be odd, whereas the right is \( 2|E| \) which is even. \(\square\)

**Problem 3.** Show that a tree on \( n \) vertices has \( n - 1 \) edges.
Solution  By strong induction on $n$. If $n = 1$ then the tree is a single vertex and has no edges. Now suppose $n > 1$ and that every tree on $m < n$ vertices has $m - 1$ edges. Let $T$ be a tree on $n$ vertices, we must show it has $n - 1$ edges. Pick any edge $e$ of $T$. Then $T - e$ is disconnected: the edge $e$ is not in any cycles, so it is the unique path joining its vertices. Furthermore, $T - e$ consists of exactly two vertex-disjoint trees, $T_1$ and $T_2$, with say $n_1$ and $n_2$ vertices each. By induction, $T_1$ has $n_1 - 1$ edges and $T_2$ has $n_2 - 1$ edges. So $T$ has $(n_1 - 1) + (n_2 - 1) + 1$ edges – the extra 1 accounts for edge $e$ which we removed. Now $n_1 + n_2 = n$, so we are done. □

Problem 4. Prove that if $G$ is a connected planar graph with $f$ faces, $e$ edges, and $n$ vertices, then

$$n - e + f = 2.$$ 

Solution  By induction on $f$. If $f = 1$, then the graph has only one face, so it must be a tree. A tree on $n$ vertices has $n - 1$ edges by the last problem, and fits the formula. Now suppose $f > 1$. Then the graph contains a cycle, $C$. If we remove an edge $e$ of that cycle, then $f$ drops by one, $n$ stays the same, and $e$ drops by 1. Now by induction, $n - (e - 1) + (f - 1) = 2$ and this gives $n - e + f = 2$. □

Problem 5. Can there be a three dimensional “perfectly symmetric” polyhedron with ten faces?

Solution  If there was such a polyhedron, we could draw it as a planar graph in which all vertices have degree $r$ and all faces have size $s$, for some values of $r$ and $s$ with $r, s \geq 3$. If there are $f$ faces, then the number of edges is $fs/2$. The number of edges is also $nr/2$, where $n$ is the number of vertices. So $nr/2 = fs/2$ which means $n = fs/r$. Now from the last result,

$$fs/r - fs/2 + f = 2.$$ 

This gives using $f = 10$,

$$10(2r + 2s - rs) = 4r$$

So

$$r = 2 + \frac{16}{5s - 8}.$$ 

For $s \geq 3$, $5s - 8$ never divides 16, so we are done. □
Problem 6. Is the complete graph on five vertices planar?

Solution If it were planar, then we would have $5 - 10 + f = 2$ so it would have 7 faces. Note that every face has size at least three. If the faces sizes are $f_1, f_2, \ldots, f_7$, then we know $f_1 + f_2 + \cdots + f_7 = 20$, twice the number of edges. But then $f_i \leq 2$ for some $i$, which is a contradiction. Solution

Problem 7. Suppose the edges of the complete graph on six vertices are colored red and blue. Prove that the graph contains a red triangle or a blue triangle.

Solution Every vertex of a complete graph of order six has degree five. Therefore three of the edges on a particular vertex $v$ must have the same color, say red. In other words, we find three red edges $\{v, x\}, \{v, y\}$ and $\{v, z\}$. If the edges $\{x, y\}$ and $\{x, z\}$ and $\{y, z\}$ are all blue, then this makes a blue triangle. So one of them, say $\{x, y\}$, must be red. However then $\{v, x\}, \{v, y\}, \{x, y\}$ make a red triangle, and we are done. □

Problem 8. Show that if a graph has $n \geq 3$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges, then it contains a triangle.

Solution By induction on $n$. For $n = 3$, the quantity is 3 so the graph in fact is a triangle. Suppose $n > 3$ and that the statement is true for all graphs on $n$ vertices. Let $G$ be a graph on $n + 1$ vertices with at least $\lfloor (n + 1)^2/4 \rfloor + 1$ edges. We want to show $G_{n+1}$ has a triangle. Delete edges from $G_{n+1}$ until it has exactly $\lfloor (n + 1)^2/4 \rfloor + 1$ edges. If this graph is $H_{n+1}$, and we can show $H_{n+1}$ has a triangle, then of course $G_{n+1}$ contains a triangle. Now we know in $H_{n+1}$ that

$$\sum_{v \in V} d(v) = 2|E| = 2\lfloor (n + 1)^2/4 \rfloor + 1.$$

Therefore (check this) there must be a vertex $v$ in $H_{n+1}$ with $d(v) \leq \lfloor (n + 1)/2 \rfloor$. Deleting this vertex and all edges on it, we get a graph $G_n$ with $n$ vertices and exactly the following number of edges:

$$\lfloor (n + 1)^2/4 \rfloor + 1 - \lfloor (n + 1)/2 \rfloor = \lfloor n^2/4 \rfloor + 1.$$

By induction, $G_n$ contains a triangle, and therefore so does $G_{n+1}$. □

Problem 9. Prove that a graph on $n$ vertices with $2n - 1$ edges has a subgraph of minimum degree at least three.
**Solution** Repeatedly remove vertices of degree at most two until there are none left, or the graph is empty. If there is no subgraph of minimum degree at least three, then after \( n - 4 \) steps we have a graph with at least
\[
2n - 1 - 2(n - 4) = 7
\]
edges, and that is impossible, since a graph on four vertices has at most six edges.

\(\square\)

5.3 Homework Problems

0. Prove that every graph can be drawn in \( \mathbb{R}^3 \) without crossing edges.

1. Prove that a graph is bipartite if and only if it has no odd cycles.

2. Let \( Q_n \) denote the graph whose vertices are binary strings of length \( n \), and where two strings form an edge in the graph if they differ in exactly one position. Determine the number of edges in \( Q_n \). Now suppose we join two strings if they differ in exactly \( k \) positions. Let this graph be \( Q^k_n \), so \( Q^1_n = Q_n \). How many edges does \( Q^k_n \) have?

3. Prove that if a graph on \( n \) vertices has no triangles and no cycles of length four, then
\[
\sum_{v \in V} d(v)^2 \leq n(n - 1).
\]
Deduce that the graph has at most \( \frac{1}{2}n\sqrt{n - 1} \) edges.

4. Show that a graph with more that \( \lfloor n^2/3 \rfloor \) edges contains \( K_4 \).

5. Prove that if a graph on \( n \) vertices has at least \( 3n/2 \) edges, then it contains an even cycle.

6* Let \( C_k \) denote a cycle of length \( k \). A straight line drawing of a graph is a drawing of the graph in the plane so that vertices are distinct points, edges are straight lines, and if two edges share no vertices, then they cross. Determine which cycles \( C_k \) have a straight line drawing. Show that if a graph on \( n \) vertices has a straight line drawing, then it has no more than \( 2n - 2 \) edges.

7. Let \( X \) be a collection of distinct points in \( \mathbb{R}^3 \). Prove that the number of pairs of points in \( X \) at distance one from each other is at most \( \frac{1}{2}n^{5/3} + n \).
8. A perfect matching in a graph is a collection of vertex-disjoint edges whose union covers all the vertices of the graph. Determine the number of perfect matchings in $K_n$ for $n \geq 2$, and then determine the number of perfect matchings in the ladder graph $L_n$ obtained by taking two disjoint paths of $n$ vertices, and joining the $i$th vertex of one path to the $i$th vertex of the other for $1 \leq i \leq n$.

9* Prove that the number of walks of length three in a graph $G = (V, E)$ is at least

$$\frac{8|E|^3}{n^2}.$$ 

For example, if we consider the graph $K_2$ with vertex set $\{u, v\}$, then the two walks of length three are

$$(u, \{u, v\}, v, \{v, u\}, u, \{u, v\}, v) \text{ and } (v, \{v, u\}, u, \{u, v\}, v, \{v, u\}, u).$$