Math 150
Practice with Chapter 1 Concepts and Calculations

1. Let \( l \) be the line through the point \((3, 2)\) with slope \(1/2\). Find the equation of the line \( l \). Simplify your answer so that it is in \( y = mx + b \) form.

Point \((3, 2)\)
Slope \(1/2\)

\[
y = \frac{1}{2}(x - 3) + 2
\]
\[
y = \frac{1}{2}x - \frac{3}{2} + 2
\]
\[
y = \frac{1}{2}x - \frac{1}{2}
\]

2. Find the vertex of the parabola \( y = x^2 + 3x + 1 \).

Compute the square:

\[
y = x^2 + 3x + \frac{9}{4} - \frac{9}{4} + 1
\]
\[
y = (x + \frac{3}{2})^2 - \frac{1}{4}
\]

Vertex: \((-\frac{3}{2}, -\frac{1}{4})\)

3. What is the largest domain where the following functions are defined?

(a) \( f(x) = \sqrt{x^2 + 1} \)

Need \( x^2 + 1 \geq 0 \)
\[
\Rightarrow x^2 \geq -1
\]
\( x^2 \) is always \( \geq -1 \)

\[\text{thus domain: } (-\infty, \infty).\]

(b) \( f(x) = \frac{x+1}{2x-1} \)

Need \( 2x - 1 \neq 0 \)
\[
\Rightarrow 2x \neq 1 \Rightarrow x \neq \frac{1}{2}
\]

Domain: \((-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)\)
(c) \( f(x) = \frac{x+7}{\sqrt{|x|-5}} \)

Need \( |x| - 5 > 0 \)
\[ |x| > 5 \]

Domain: \((-\infty, -5) \cup (5, \infty)\)

4. Let \( f(x) \) be the function pictured in the figure. The domain is \([0,6]\).

(a) What is the range of \( f(x) \)?

\([-1, 1]\)

(b) What is the domain and range for \( 2f(x) + 3 \)? Describe the transformations in words. Sketch this function on the axis with the graph of \( f \).

Vertical stretch of 2, up 3

Domain: \([0, 6]\)

Range: \([1, 5]\)
(c) What is the domain and range of $f(-2x)$? Describe the transformations in words. Sketch this function on the axis with the graph of $f$.

Domain: $[0, -3]$
Range: $[-1, 1]$

Horizontal shrink of 2
Flip over y axis

(d) What is the domain and range of $3f(x-2)$? Describe the transformations in words. Sketch this function on the axis with the graph of $f$.

Domain: $[2, 8]$
Range: $[-3, 3]$

Right 3, vertical stretch of 3

5. Graph $y = x^3$ and $y = 3(x + 1)^3$ on the same axis.

6. Assume $g$ is an even function and $g(x)$ has the values below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
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<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

Since $g$ is even, $g(-x) = g(x)$ thus

<table>
<thead>
<tr>
<th>$x$</th>
<th>$g(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td>-4</td>
<td>1</td>
</tr>
<tr>
<td>-5</td>
<td>4</td>
</tr>
</tbody>
</table>

(a) What is $g(2) + g(3)$?

$3 + 4 = 7$
(b) What is \( g(-2)+g(-3) \)?

\[ 3 + 4 = 7 \]

(c) What is \( 2g(-4) \)?

\[ 2(-1) = 2 \]

(d) What is \(-g(2)\)?

\[ -3 \]

7. Give an example of a function with domain 1, 2, 3, 4 and range 2, 3, 5. You may present the function in a table. Is the function you presented one-to-one? Does it have an inverse?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
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<tr>
<td>2</td>
<td>3</td>
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<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

This function is not one-to-one since \( f(2) = f(3) \). Since it is not one-to-one, the function does not have an inverse.
8. Let \( f(x) = x^3 + 1 \), \( g(x) = \frac{x + 1}{x - 1} \), \( h(x) = (x - 1)^{1/3} \)

(a) What is \((g \circ f)(x)\)?

\[
g(x^3 + 1) = \frac{x^3 + 1}{x^3}
\]

(b) What is \((h \circ g \circ f)(1)\)?

\[
f(1) = 2
\]
\[
g(2) = \frac{3}{1} = 3
\]
\[
h(3) = 2^{1/3} = 3^{\sqrt[3]{2}}
\]

(c) What is \((f \circ h)(x)\)?

\[
h \circ f \left( (x-1)^{1/3} \right) = x - 1 + 1 = x
\]

(d) What is \((h \circ f)(x)\)?

\[
h \left( x^3 + 1 \right) = \left( x^3 + 1 - 1 \right)^{1/3} = x
\]
(e) Find $f^{-1}(x)$.

Since $h(x) = h \circ f(x) = x$

$h = f^{-1}(x)$

9. Write $h(x) = \frac{\sqrt{x+1}+7}{(x+1)^{3/2}-1}$ as a composition of simpler functions, i.e. find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$.

$g(x) = (x+1)^{1/2}$

$f(x) = \frac{x+7}{x^3-1}$

10. Find the largest interval containing $x = 2$ where $f(x) = 2x^2 - 4x + 5$ has an inverse.

Graph is parabola

$2^{x^2 - 4x + 5}$

One-to-one on $(\infty, 1)$ and $(1, \infty)$

$2(x^2 - 2x + 1 - 1 + 5/2)$

$(\infty, \infty)$ contains $2$

Vertex is: $(1, 3)$

Thus $(1, \infty)$ is largest domain when $f(x)$ has inverse.

11. Consider the function $f(x) = x^3 + 3$ on $[0, 3]$.

(a) Use the fact that $f(x)$ is increasing on the interval to find the range of $f(x)$.

Since $f(x)$ is increasing, we can find the range by finding the endpoints which occur at the endpoints of the domain.

$f(0) = 3$ $f(3) = 30$

Range: $[3, 30]$
(b) Find \( f^{-1}(y) \).

\[
\begin{align*}
y &= x^3 + 3 \\
y - 3 &= x^3 \\
x &= \sqrt[3]{y - 3} \\
f^{-1}(y) &= 3\sqrt[3]{y - 3}.
\end{align*}
\]

(c) What is the domain of \( f^{-1}(y) \)? (Hint: Do NOT use part (b) to help you.)

\[
\text{domain of } f^{-1}(y) = \text{range of } f(x)
\]

\[
= [3, 30]
\]

(d) What is the range of \( f^{-1}(y) \)?

\[
\text{range of } f^{-1}(y) = \text{domain of } f(x)
\]

\[
= [0, 3]
\]

12. Find the inverse of \( f(x) = \frac{2x}{x + 3} \)

\[
\begin{align*}
y &= \frac{2x}{x + 3} \\
\text{Solve for } x: \\
y(x + 3) &= 2x \\
yx + 3y &= 2x \\
yx - 2x &= 3y \\
x(y - 2) &= 3y \\
x &= \frac{3y}{y - 2}
\end{align*}
\]

\[
\therefore \quad f^{-1}(x) = \frac{3x}{x - 2}
\]
13. Consider the function in the table below. It tells you for each element of the domain \{newborns, infants, toddlers, kids, teens, adults\} how much sleep is needed in hours.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>newborns</td>
<td>17</td>
</tr>
<tr>
<td>infants</td>
<td>14</td>
</tr>
<tr>
<td>toddlers</td>
<td>12</td>
</tr>
<tr>
<td>kids</td>
<td>11</td>
</tr>
<tr>
<td>teens</td>
<td>9</td>
</tr>
<tr>
<td>adults</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) What is \( f(\text{teens}) \)? Write a short sentence describing in words what this means.

\[ f(\text{teens}) = 9. \]
Teen need 9 hours of sleep.

(b) What is \( f^{-1}(12) \)? Write a short sentence describing in words what this means.

\[ f^{-1}(12) = \text{toddler}s. \]
The age group that need 12 hours of sleep is toddlers.

14. Below is the graph of \( f(x) = \ln(x) \). The function \( y = e^x \) is the inverse of \( \ln(x) \).

(a) Using the graph of \( y = \ln(x) \), sketch the graph of \( e^x \) on the same axis.

(b) The domain of \( \ln(x) \) is \((0, \infty)\) and the range is \((-\infty, \infty)\). What is the domain and range of \( e^x \)?

Domain: \((\infty, \infty)\)
Range: \((0, \infty)\)
15. Explain why the horizontal line test works to test if a function is one-to-one.

When we draw a horizontal line, by looking at the intersections of the line with the graph, we can see how many points have the same y value. If any horizontal line touches the graph more than once, the function is not one-to-one.

16. Simplify $2^{4}4^{15}16^{8}$ as a power of 4.

\[
\begin{align*}
(2^2)^2 &\quad 4^{15} \quad (4^2)^8 \\
4^2 &\quad 4^{15} \quad 4^{16} \\
= &\quad 4^{33}
\end{align*}
\]

17. Evaluate/Simplify the following (without a calculator):

(a) $(-8)^{5/3}$

\[
\left((-8)^3\right)^{5} = (-2)^5 = -32
\]

(b) $(2 - \sqrt{x})^2$

\[
(2 - \sqrt{x})(2 - \sqrt{x}) = 4 - 4\sqrt{x} + x
\]
18. Let \( f(x) = 2x^2 + 1 \) defined on \([0, \infty)\).

(a) Sketch the graph of \( f(x) \).

(b) Find \( f^{-1} \).

\[
y = 2x^2 + 1
\]

\[
2x^2 = y - 1
\]

\[
x^2 = \frac{y - 1}{2}
\]

\[
x = \pm \sqrt{\frac{y - 1}{2}}
\]

\[
f^{-1}(x) = \sqrt{\frac{x - 1}{2}}
\]

(c) Sketch the graph of \( f^{-1} \). (Use part (a) as a guide.)

19. Evaluate the following without a calculator:

(a) \( \log_2(8^{4.5}) \)

\[
\log_2 8^{4.5} = 4.5 \log_2 8 = (4.5)(3) = 13.5
\]
(b) $\log_9 27$

\[
\log_9 27 = x \\
9^x = 27 \\
3^{2x} = 3^3
\]

\[
2x = 3 \\
x = \frac{3}{2}
\]

(c) $\log 100000$

\[
\log 100000 = x \\
10^x = 100,000 \\
x = 5
\]

20. Solve for $x$.

\[
\log_3(2x + 1) = 81
\]

\[
3^{81} = 2x + 1 \\
2x = 3^{81} - 1 \\
x = \frac{3^{81} - 1}{2}
\]

21. Neon-18 has a half life of 2 seconds. How much time will it take to have $1/16$ of the amount of Neon-18 that you started with?

\[
\begin{align*}
2 \text{ seconds} & \rightarrow \frac{1}{2} \\
4 \text{ seconds} & \rightarrow \frac{1}{4} \\
6 \text{ seconds} & \rightarrow \frac{1}{8} \\
8 \text{ seconds} & \rightarrow \frac{1}{16}
\end{align*}
\]
22. Solve for x.

\[ \log_3(x+5) + \log_3(x-1) = 4 \]

\[ \log_3\left(\frac{x+5}{x-1}\right) = 4 \]

\[ 3^4 = (x+5)(x-1) \]

\[ 81 = x^2 + 4x - 5 \]

\[ x^2 + 4x - 86 = 0 \]

\[ x = \frac{-4 \pm \sqrt{16 + 4 \times 86}}{2} \]

\[ = \frac{-4 \pm \sqrt{360}}{2} = -2 \pm 3\sqrt{10} \]

23. Simplify \(2^{3\log_2(4) + 2\log_2(16)}\)

\[ 2^{3\log_2(4) + 2\log_2(16)} = 2^{3 \cdot 2 \cdot 4^{1/2}} \]

\[ = 2^{3 \cdot 16^2} = 4^{3 \cdot 16^2} \]

24. Express \(\log_{100} 1000\) as a quotient of common logs and solve.

\[ \log_{100} 1000 = \frac{\log_{10} 1000}{\log_{10} 100} = \frac{3}{2} \]

25. Consider the function \(f(x) = c \cdot b^{kx}\). What is the slope of the line represented by \(\log_b f(x)\)

\[ \log_b f(x) = \log_b c \cdot b^{kx} \]

\[ = \log_b c + \log_b b^{kx} \]

\[ = \log_b c + kx \]

This is a line with slope \(k\).
26. The population of Delaware in 1980 was 6 hundred thousand. In 2010 it was 900,000 (9 hundred thousand). Assume the population growth is exponential of the form \( f(x) = c \cdot b^x \). For simplicity, express the function is hundred thousands. (Use 6 and 9 as the values.)

(a) What two conditions are satisfied by \( f(x) \).

\[
\begin{align*}
f(1980) &= 6 \\
f(2010) &= 9
\end{align*}
\]

(b) Solve for \( c \) and \( b \).

\[
\begin{align*}
\text{(2) } f(1980) &= c \cdot b^{1980} = 6 \\
\text{(1) } f(2010) &= c \cdot b^{2010} = 9 \\
\text{divide (2) \text{ by (1)}} &
\end{align*}
\]

\[
\begin{align*}
\frac{b^{2010-1980}}{b^{30}} &= \frac{9}{6} \\
b &= \frac{3}{2} \\
\sqrt[30]{\frac{3}{2}} &= \frac{3}{2}
\end{align*}
\]

\[
\begin{align*}
c \left(3^0 \sqrt[30]{\frac{3}{2}}\right)^{2010} &= 9 \\
c &= \frac{9}{\left(\frac{3}{2}\right)^{67}}
\end{align*}
\]

(c) What will the population be in 2020? You do not need to simplify.

\[
f(2020) = \frac{9}{\left(\frac{3}{2}\right)^{201/3}} \cdot \left(3^0 \sqrt[30]{\frac{3}{2}}\right)^{2020}
\]

(d) When will the population reach 1,000,000. (Hint: 1,000,000=10 hundred thousand.)

\[
\begin{align*}
\frac{9}{\left(\frac{3}{2}\right)^{67}} \cdot \left(3^0 \sqrt[30]{\frac{3}{2}}\right)^x &= 10 \\
\left(3^0 \sqrt[30]{\frac{3}{2}}\right)^x &= \frac{10}{9} \left(\frac{3}{2}\right)^{67} \\
x &= \log_{3^0 \sqrt[30]{\frac{3}{2}}} \left(\frac{10}{9} \left(\frac{3}{2}\right)^{67}\right)
\end{align*}
\]
27. Simplify \( \ln\frac{2x+1}{x-1} = 2 \)

\[
\ln \frac{2x+1}{x-1} = 2
\]

\[
\frac{2x+1}{x-1} = e^2
\]

\[
2x+1 = e^2x - e^2
\]

\[
e^2x - 2x = 1 + e^2
\]

\[
x(e^2 - 2) = 1 + e^2
\]

\[
x = \frac{1 + e^2}{e^2 - 2}
\]

28. Solve for \( x \). \( e^{2x} - 4e^x = 12 \)

Let \( y = e^x \), we have

\[
y^2 - 4y = 12 = 0
\]

\[
(y + 2)(y - 6) = 0
\]

\[
y = -2, y = 6
\]

So \( e^x = -2 \) or \( e^x = 6 \)

\[
e^x = -2 \quad e^x = 6 \text{ when } x = \ln 6
\]

Practice with Chapter 2 Concepts and Calculations Sections 2.1-2.3

29. Let \( f(x) = x^2 \).

(a) Draw a graph of \( f(x) \) for \( x \) between -3 and 3.

(b) What is the slope of the secant line from \( x = 1 \) to \( x = 2 \)? Draw this secant line on the graph.

\[
\text{Slope of secant line } = \frac{f(2) - f(1)}{2 - 1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3
\]
(c) What is the slope of the secant line from $x = 1$ to $x = 1 + h$? (You should have $h$'s in your answer.)

\[
\frac{f(1+h) - f(1)}{h} = \frac{(1+h)^2 - 1}{h} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h}
\]

(d) Find the slope of the tangent line to $f(x) = x^2$ at $x = 1$ by taking the limit of the slopes of the secant lines in part (c) as $h \to 0$.

\[
\lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} 2 + h = 2
\]

30. Draw two tangent lines to the function graphed below, one at $x = 2$ and the other at $x = 4.5$. 
31. The position of a car at time $t$ is given by $f(t) = 3t^3 + 2t$.

(a) What is the average speed between $t = 1$ and $t = 2$?

$$\text{avg speed} = \frac{f(2) - f(1)}{2 - 1} = \frac{(3 \cdot 8 + 4) - (3 + 2)}{1} = \frac{25 - 5}{1} = 20$$

(b) How would you find the speed at $t = 1$?

Find the average velocity over smaller and smaller intervals,

\[ \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \]
32. Using the graph below, evaluate the following limits or function values or state that they do not exist.

\[ y = f(x) \]

(a) \( \lim_{x \to 1^+} f(x) = 2 \)
(b) \( \lim_{x \to 1^-} f(x) = 2 \)
(c) \( \lim_{x \to 1} f(x) = 2 \)
(d) \( f(1) = 3 \)
(e) \( \lim_{x \to 2^+} f(x) = 2 \)
(f) \( \lim_{x \to 2^-} f(x) = 1 \)
(g) \( \lim_{x \to 2} f(x) = \text{DNE} \)
(h) \( f(2) = 1 \)
(i) \( \lim_{x \to 3^+} f(x) = 3 \)
(j) \( \lim_{x \to 3^-} f(x) = 3 \)
(k) \( \lim_{x \to 3} f(x) = 3 \)
(1) \( f(3) \) undefined

(m) On what intervals is the function continuous?

\[ (0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \]

33. Draw a graph of a function defined on the domain \([-2, 4]\) where \( \lim_{x \to 2} = 4, f(2) = 3 \) and the function is continuous everywhere except \( x = 2 \) and \( x = 3 \).

34. Is it possible to have a continuous function (continuous at all \( x \) values) where \( \lim_{x \to 1} = 2, f(1) = -1 \)? If so, provide an example (via graph).

If \( \lim_{x \to 1} f(x) = 2 \) and \( f(1) = -1 \) then

the function is not continuous, thus

it is not possible.

35. Evaluate:

\[
\lim_{x \to 3^-} \frac{|x-3|}{x-3}
\]

If \( x < 3 \), \( x - 3 < 0 \) so \( |x-3| = -(x-3) \)

\[
\lim_{x \to 3^-} \frac{|x-3|}{x-3} = \lim_{x \to 3^-} \frac{-(x-3)}{x-3} = \lim_{x \to 3^-} -1 = -1
\]
36. Assume 
\[ \lim_{x \to 4} f(x) = 2 \text{ and } \lim_{x \to 4} g(x) = -1. \]

Evaluate the following. Justify each step by indicating the appropriate limit law.

(a) 
\[ \lim_{x \to 4} 5f(x) - 3g(x) = 5 \lim_{x \to 4} f(x) - 3 \lim_{x \to 4} g(x) \]

by subtraction law

\[ = 5 \cdot 2 - 3(-1) = 10 + 3 = 13 \]

(b) 
\[ \lim_{x \to 4} 3f(x)^2 - \frac{f(x)}{g(x)} = 3 \left( \lim_{x \to 4} f(x) \right)^2 - \lim_{x \to 4} \frac{f(x)}{g(x)} \]

by quotient rule

\[ = 3(2)^2 - \frac{\lim_{x \to 4} f(x)}{\lim_{x \to 4} g(x)} \]

\[ = 3 \cdot 4 - \frac{2}{-1} = 12 + 2 = 14 \]

37. Evaluate the following limits:

(a) 
\[ \lim_{x \to 2} \frac{x^2 - x - 6}{x - 3} \]

2 is in the domain so

\[ \lim_{x \to 2} \frac{x^2 - x - 6}{x - 3} = \frac{4 - 2 - 6}{1} = \frac{-4}{1} = -4 \]
(b) \[ \lim_{x \to 3} \frac{x^2 - x - 3}{x - 3} = \lim_{x \to 3} \frac{(x+2)(x-3)}{x-3} = \lim_{x \to 3} x + 2 = 5 \]

(c) \[ \lim_{h \to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h \to 0} \frac{h(3+3h+h^2)}{h} = \lim_{h \to 0} 3+3h+h^2 = 3 \]

(d) \[ \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} = \lim_{h \to 0} \frac{(\sqrt{1+h} + 1)}{(\sqrt{1+h} + 1)} - \lim_{h \to 0} \frac{h}{h(1+h+1)} = \lim_{h \to 0} \frac{1}{\sqrt{1+h} + 1} = \frac{1}{2} \]

(e) \[ \lim_{t \to 0} \frac{t+1}{t(t+1)} - \frac{1}{t^2 + t} = \lim_{t \to 0} \frac{t}{t^2(t+1)} = \lim_{t \to 0} \frac{1}{t+1} = 1 \]
38. (a) Using the fact that \(-1 \leq \cos \frac{1}{x} \leq 1\), what bounds can there on \(|\cos \frac{1}{x}|)?

\[
0 \leq |\cos \frac{1}{x}| \leq 1
\]

(b) Using part (a) to find bounds on \(|x||\cos \frac{1}{x}|)?

by multiplying by \(1x\) (which is greater than 0)

\[
10 \leq 1\times|\cos \frac{1}{x}| \leq 1\times1
\]

(c) What is \(\lim_{x \to 0} 0\)?

\[
\lim_{x \to 0} 0 = 0
\]

(d) What is \(\lim_{x \to 0} |x|\)?

\[
\lim_{x \to 0} |x| = 0
\]

(e) Use (c) and (d) to find \(\lim_{x \to 0} |x||\cos(\frac{1}{x})|)? What theorem did you use?

Since \(|x||\cos \frac{1}{x}|\) is between 0 and \(1x\) and \(\lim_{x \to 0} 0 = \lim_{x \to 0} 1x = 0\), then \(\lim_{x \to 0} |x||\cos \frac{1}{x}| = 0\) by Squeeze theorem.