Homework #2

1. Prove $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$.

2. Find $C_1, C_2 > 0$ such that
   
   (a) $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$.
   
   (b) $C_1 \|x\|_2 \leq \|x\|_\infty \leq C_2 \|x\|_2$.

3. Suppose we define the norm $\|\cdot\|$ to be equivalent to the norm $\|\cdot\|'$ in a vector space $V$ if and only if there exists real numbers $C_1, C_2 > 0$ such that $C_1 \|v\| \leq \|v\|' \leq C_2 \|v\|$ for all $v \in V$.

   (a) Show $\|\cdot\|$ equivalent to $\|\cdot\|'$ implies $\|\cdot\|'$ equivalent to $\|\cdot\|$.

   (b) Show $\|\cdot\|$ equivalent to $\|\cdot\|'$ and $\|\cdot\|'$ equivalent to $\hat{\|\cdot\|}$ implies $\|\cdot\|$ equivalent to $\hat{\|\cdot\|}$.

   (c) Let $V$ be the set of $n$-component real vectors and $W$ be the set of $n \times n$ real matrices. Show $\|\cdot\|$ equivalent to $\|\cdot\|'$ in $V$ implies the induced norms are equivalent in $W$.

4. In Matlab, set up $n \times n$ Hilbert matrices with $n = 3, 6, 9$. using “hilb”.

   (a) Indicate, using Matlab, that the matrices are symmetric and positive definite (a little more evident looking at eigenvalues rather than determinants).

   (b) Use “cond” to calculate the condition numbers $K_1, K_2, K_\infty$ for each matrix. Are they looking well-conditioned?

5. Let $x + \delta x$ be an approximate solution to $Ax = b$. Define the residual $r = b - A(x + \delta x)$.

   (a) Show $(A + \delta A)(x + \delta x) = b$ is satisfied with $\delta A = \|x + \delta x\|_2^{-2} r(x + \delta x)^T$.

   (b) Show $A(x + \delta x) = b + \delta b$ is satisfied with $\delta b = -r$.

   (c) If the relative residual (defined by $\|r\|/\|b\|$) is small under a chosen norm, what additional conditions are needed to guarantee a small relative error for the approximate solution? If the relative residual is large, is the relative error also forced to be large?

6. Consider the linear system $Ax = b$ and the perturbed linear system $A(x + \delta x) = b + \delta b$ with

   \[ A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}. \]

   (a) Compute $A^{-1}$ and find $K_\infty(A)$. 


(b) Find vectors $b$ and $\delta b$, satisfying $||\delta b||_\infty/||b||_\infty = C \cdot 10^{-5}$ for some real number $1 \leq C < 10$, such that

$$\frac{||\delta x||_\infty}{||x||_\infty} = K_\infty(A) \frac{||\delta b||_\infty}{||b||_\infty}.$$ 

(c) Thus, what is the relative error of the approximation $x + \delta x$ as a percentage?

7. Prove

$$\frac{||\delta b||}{||b||} \leq K(A) \frac{||\delta x||}{||x||}$$

when $Ax = b$ and $A(x + \delta x) = b + \delta b$. Also determine when equality holds.

8. Prove

$$\frac{||\delta x||}{||x + \delta x||} \leq K(A) \left( \frac{||\delta A||}{||A||} + \frac{||\delta b||}{||b + \delta b||} + \frac{||\delta A||}{||A||} \frac{||\delta b||}{||b + \delta b||} \right)$$

where $Ax = b$ and $(A + \delta A)(x + \delta x) = b + \delta b$ with $b + \delta b \neq 0$ (Hint: it helps to first show

$$||\delta x|| \leq ||A^{-1}||(||\delta b|| + ||\delta A||)||x + \delta x||$$

$$||b + \delta b|| \leq (||A|| + ||\delta A||)||x + \delta x||$$

and use these to get the result).