1. Given a real number $x = \pm 0.x_1x_2x_3 \ldots \cdot 10^r$ and a machine that only stores $n$ digits in the mantissa, suppose $f_{\text{chop}}(x)$ is computed by chopping of the mantissa rather than the usual rounding: $f_{\text{chop}}(x) = \pm 0.x_1x_2 \ldots x_n \cdot 10^r$. Find a constant that bounds the relative error of this $f_{\text{chop}}(x)$ for all $x \neq 0$. How does it compare to the relative error using the standard $f_l(x)$?

2. Let $x$ and $y$ be real numbers. For the following operations, find the relative error of the machine version in terms of $x,y,$ and relative differences of the form $(f_l(z) - z)/z$ for $z$ a real number. Is the relative error small for all $x,y$ when the unit roundoff error is small?
   
   (a) $x + y$.
   (b) $x \cdot y$.
   (c) $x/y$ when $y \neq 0$.

3. Let $u$ be the unit roundoff error.
   
   (a) Show the machine version of $((x_1 + x_2) + x_3) + x_4$ can be written as $\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{x}_4$, where $\tilde{x}_i = x_i(1 + \epsilon_i)$ with $|\epsilon_i| \leq 4u + \mathcal{O}(u^2)$.
   
   (b) Show the machine version of $(x_1 + x_2) + (x_3 + x_4)$ can be written as $\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3 + \tilde{y}_4$, where $\tilde{y}_i = x_i(1 + \delta_i)$ with $|\delta_i| \leq 3u + \mathcal{O}(u^2)$.
   
   (c) Compare the bounds on $\epsilon_i$ and $\delta_i$ when computing $\sum_{i=1}^{2^n} x_i$ using the two strategies above. Which is better? To clarify, the second strategy for 8 numbers performs $((x_1 + x_2) + (x_3 + x_4)) + ((x_5 + x_6) + (x_7 + x_8))$.

4. Consider the linear system $Ax = b$, where $A$ is an upper triangular $n \times n$ matrix of real numbers and $b$ is an $n$-component vector of real numbers. Show that back substitution on this linear system performed in the machine leads to the solution $x + \delta x$ satisfying $(A + \delta A)(x + \delta x) = b$ with
   
   $$|\delta A| \leq C(n)u|A| + \mathcal{O}(u^2).$$

   Find an explicit form for $C(n)$ and compare it to the case where $A$ and $b$ are already given as machine numbers.

5. The $LU$ factorization of a matrix real-valued matrix $A$ is computed through the formulas
   
   $l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik}u_{kj}}{u_{jj}},$

   for $i > j$, and

   $u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik}u_{kj},$
for $i \leq j$. In the machine, $\hat{L} = (\hat{l}_{ij})$ and $\hat{U} = (\hat{u}_{ij})$ are computed instead. Show $A + E = \hat{L}\hat{U}$ satisfies

$$|e_{ij}| \leq C(n)u \sum_{k=1}^{j} |\hat{l}_{ik}||\hat{u}_{kj}| + O(u^2),$$

for some $C(n)$, where $E = (e_{ij})$. Find an explicit form for $C(n)$. (Hint: treat the cases of $i > j$ and $i \leq j$ individually)

6. Now show the approximate solution $x + \delta x$ obtained when solving $Ax = b$, for real-valued $A, b$, in the machine by $LU$ factorization followed by forward and back substitution satisfies $(A + \delta A)(x + \delta x) = b$ where

$$|\delta A| \leq C(n)u|\hat{L}||\hat{U}| + O(u^2)$$

for some $C(n)$. Find an explicit form for $C(n)$. 