1. Prove a tridiagonal matrix with nonzero diagonal elements satisfies the property: \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for some \( 0 \neq \alpha \in \mathbb{C} \) if and only if \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for all \( 0 \neq \alpha \in \mathbb{C} \).

2. Suppose \( A \) is nonsingular, has nonzero diagonal elements, and satisfies: \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for some \( 0 \neq \alpha \in \mathbb{C} \) if and only if \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for all \( 0 \neq \alpha \in \mathbb{C} \). Prove \( \rho(B_{GS}) = \rho(B_J)^2 \).

3. (a) Let \( r_1, r_2 \in \mathbb{C} \) be the roots of the quadratic polynomial \( x^2 - bx + c \), where \( b, c \) are real. Prove \( |r_1| < 1 \) and \( |r_2| < 1 \) if and only if \( |c| < 1 \) and \( 1 + c - |b| > 0 \).

(b) Suppose \( A \) is nonsingular, has nonzero diagonal elements, and satisfies: \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for some \( 0 \neq \alpha \in \mathbb{C} \) if and only if \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for all \( 0 \neq \alpha \in \mathbb{C} \). Suppose also \( B_J \) has real eigenvalues. Prove SOR is convergent if and only if \( 0 < \omega < 2 \) and \( \rho(B_J) < 1 \).

4. Suppose \( A \) is nonsingular, has nonzero diagonal elements, and satisfies: \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for some \( 0 \neq \alpha \in \mathbb{C} \) if and only if \( \det(\alpha D^{-1}E + \alpha^{-1}D^{-1}F - \lambda I) = 0 \) for all \( 0 \neq \alpha \in \mathbb{C} \). Suppose also \( B_J \) has real eigenvalues and \( \rho(B_J) < 1 \). Prove the optimal \( \omega \) (giving the fastest convergence) in SOR is

\[
\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B_J)^2}}.
\]

5. Consider Richardson’s iteration:

\[
x^{(k+1)} = (I - \alpha A)x^{(k)} + \alpha b
\]

for \( \alpha \in \mathbb{R} \) and assume \( A \) has real and positive eigenvalues.

(a) Prove the iterative method converges for all initial guesses if and only if \( 0 < \alpha < \frac{2}{\lambda_{max}} \), where \( \lambda_{max} \) is the largest eigenvalue of \( A \).

(b) Prove the optimal \( \alpha \) is

\[
\alpha_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}},
\]

where \( \lambda_{min} \) is the smallest eigenvalue of \( A \). What is the spectral radius in this case?