Homework #9

1. Given $A$ and $v \neq 0$, let $s$ be the minimum degree of a monic polynomial $p$ satisfying $p(A)v = 0$.

   (a) Prove $\dim K_m(A; v) = m$ when $m \leq s$.
   (b) Prove $K_m(A; v) = K_s(A; v)$ when $m > s$.

2. Let $Ax = b$ be the linear system of interest.

   (a) Show, in the Arnoldi algorithm, if $w_k = 0$ then $\{v_1, \ldots, v_k\}$ is an orthonormal basis for $K_m(A; v)$ for any $m \geq k$.
   (b) Count the number of flops needed to compute $V_k$ using the Arnoldi algorithm.
   (c) How many elements does the matrix $V_k$ have? Do we expect $V_k$ to be sparse if $A$ is sparse?

3. (a) Show if $v_1, \ldots, v_m$ is the Arnoldi algorithm computed orthonormal basis for $K_m(A; v)$, then $AV_m = V_mH_m + w_m e_m^T$.
   (b) In addition, if the Arnoldi method continues without breakdown to compute $v_{m+1}$, show $AV_m = V_{m+1}H_m$.

4. (a) Show the Arnoldi method for $n \times n$ linear systems converges in $\leq n$ steps.
   (b) Show GMRES for $n \times n$ linear systems converges in $\leq n$ steps.

5. Count the number of flops needed to solve $Hz = b$ by LU factorization, where $H$ is upper Hessenberg and $b_i = 0$ for $i \neq 1$. How is this important to the Arnoldi method?

6. The least squares problem minimizes $||b - Ay||_2$, where $A$ is an $n \times m$ matrix.

   (a) Explain how minimization of $||b - Ay||_2$ gives the normal equations for a least squares problem.
   (b) Explain how QR factorization can be used to solve a least squares problem.
   (c) Explain how SVD can be used to solve a least squares problem.