Math 20E
First Midterm Examination
July 11, 2013

Turn off and put away your cell phone.
No calculators or any other devices are allowed on this exam.
You may use one 8.5 x 11 page of handwritten notes, but no other assistance on this exam.
Read each question carefully, answer each question completely, and show all of your work.
Unsupported answers earn no credit.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

Solutions

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1. (a) (7 points) Find an equation for the tangent plane to the graph \( z = \sin(x) - \cos^2(y) \) at the point \((\pi/2, \pi)\).

\[
\begin{align*}
&f(x, y) = \sin(x) + \cos^2(y) \\
&f_x(x, y) = \cos(x) \\
&f_y(x, y) = 2\cos(y)(-\sin(y)) = -2\cos(y)\sin(y) \\
\text{tangent plane is} \quad z &= 2 + C(x - \pi/2) + C(y - \pi) \quad \text{or} \quad z = 2
\end{align*}
\]

(b) (8 points) Approximate \(\sin(1.5) - \cos^2(3.1)\) using a second order Taylor approximation. If it is relevant, you may refer to work from part (a). You may leave your answer unsimplified.

\[
\begin{align*}
&f_{xx} = -\sin(x) \\
&f_{xy} = 0 \\
&f_{yy} = -2(-\sin(y))\cos(y) - 2\cos(y)\sin(y) = 2\sin^2(y) - 2\cos^2(y) \\
\text{using the linear terms from part (a),} \\
T_2(x, y) &= 2 + C(x - \pi/2) + C(y - \pi) \approx -1(x - \pi/2)^2 \frac{1}{2} + O(x - \pi/2)(y - \pi) \\
&= -2 \frac{(y - \pi)^2}{2} \\
&= 2 - \frac{1}{2} (x - \pi/2)^2 - (y - \pi)^2 \\
f(1.5, 3.1) &\approx T_2(1.5, 3.1) = 2 - \frac{1}{2} (1.5 - \pi/2)^2 - (3.1 - \pi)^2
\end{align*}
\]
2. (12 points) For each of the following questions, circle T if the statement is true, and F if the statement is false, and give a brief explanation if it is false.

T 0 \int_0^1 \int_0^1 (x^2 + 2y)dydx = \int_0^1 \int_0^1 (x^2 + 2y)dydx

\int_0^1 \int_0^1 x(x^2 + 2y)dydx = \int_0^1 \int_0^1 (x^2 + 2y)dydx

should not have constants on outer integral

T 0 (A - B)(A + B) = A^2 + A B - B A + B^2

A B \neq B A \text{ in general}

T 0 A\text{ differentiable function } f(x, y) \text{ is also continuous.}

This is why we required differentiability to include the "looks like a tangent plane" condition.

(Explanation was not required.)

T 0 \text{ The integral } \int_0^1 \int_0^1 e^{x^2 + y^2} dydx \text{ describes the volume under the graph}

z = e^{x^2 + y^2} \text{ over the region in } x^2 \text{ bounded by } x = y, x = 1, \text{ and } y = 0.

\int_0^1 \int_0^1 e^{x^2 + y^2} dydx \text{ is the integral of } e^{x^2 + y^2}

over the unit square

not the region

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3. Let \( f(x, y) = xy \), and let \( D \) be the region in \( \mathbb{R}^2 \) bounded by \( y = x^2 \) and \( y = x \).

(a) (3 points) Sketch \( D \), labeling each boundary with the appropriate equation.

(b) Set up an integral for \( f(x, y) \) in both orders:

i. (5 points) \( \iint_D f(x, y) \, dy \, dx \)

\[
\int_0^1 \int_{x^2}^x xy \, dy \, dx
\]

ii. (5 points) \( \iint_D f(x, y) \, dx \, dy \)

\[
\int_0^1 \int_y^{x^2} x y \, dx \, dy
\]

This problem is continued on the next page.
(c) (3 points) Evaluate \( \iiint f(x, y, z) \, dx \, dy \, dz \).

\[
\begin{align*}
&\int \int \int_{x^2} x y \, dy \, dx \\
&= \int_{0}^{1} \left[ \frac{1}{2} x y^2 \right]_{x^2}^{1} \, dx \\
&= \int_{0}^{1} \frac{1}{2} x - \frac{1}{2} x (x^2)^2 \, dx \\
&= \int_{0}^{1} \frac{1}{2} x^3 - \frac{1}{2} x \, dx \\
&= \left[ \frac{1}{2} \left( \frac{1}{4} x^4 - \frac{1}{6} x^6 \right) \right]_{0}^{1} \\
&= \frac{1}{2} \left( \frac{1}{4} - \frac{1}{6} \right) \\
&= \frac{1}{2} \left( \frac{3}{12} - \frac{2}{12} \right) \\
&= \frac{1}{24}
\end{align*}
\]
1. Let \( W \) be the region bounded by the sphere of radius 2 centered at the origin, and the cone \( z = \sqrt{x^2 + y^2} \).

(a) (3 points) Sketch \( W \) and label each surface with an appropriate equation.

\[ x^2 + y^2 + z^2 = 4 \]

(b) (2 points) Is it possible to find the volume of \( W \) using a single integral (not a sum of integrals) in the order \( dx \, dy \, dz \)? Give a brief explanation.

No. A cross-section looks like this:

We would have to split it into 2 integrals because the boundaries have different limits than the bottom.

(c) (10 points) Set up a triple integral for the volume of \( W \) in rectangular coordinates. You do not need to evaluate it.

\[
\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{2} dz \, dy \, dx
\]

(In the order of integration, you would have)

\[
\int_{-\sqrt{2}}^{\sqrt{2}} \left[ \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left( \int_{\sqrt{x^2+y^2}}^{2} 1 \, dz \right) \right] \, dy \, dx
\]

\[ z^2 + y^2 = x^2 \quad \text{and} \]

radius is that of the circle of intersection

\[
x^2 + y^2 + z^2 = 4, \quad z = \sqrt{2-x^2-y^2} \]

\[ 2z^2 = 4 \]

\[ z = \pm \sqrt{2} \]

\[ \sqrt{x^2+y^2} \geq 0 \quad \text{and} \quad z = \sqrt{2} \]
5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(u, v) = (2u + 3v, 4u + 7v).$$

(a) (2 points) Is $T$ linear? You may assume the correspondence between linear functions and matrices.

$$T(u, v) = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Since $T$ can be written as a matrix times the vector $(u, v)$, $T$ is linear.

(b) (4 points) Is $T$ one-to-one and onto (a bijection)? If you use part (a), be sure to state that you are doing so and why.

Since $T$ is linear, $T$ is a bijection if and only if $\det(\mathcal{D}T) \neq 0$.

$$\det(\mathcal{D}T) = \begin{vmatrix} 2 & 3 \\ 4 & -7 \end{vmatrix} = -14 - 12 = -26 \neq 0$$

Thus $T$ is linear a bijection.