Problem 1(6.5.4)

This question might not fit to the definition of generalized likelihood ratio exactly, but by Neyman Pearson Lemma from lecture, we know that the likelihood ratio test (also Uniform Power Test) is that we reject \( H_0 \) when

\[
\Lambda = \frac{L(\mu_0)}{L(\mu_1)} \leq k
\]

And \( k \) is the key in this critical region. To find \( k \), we make Type I error be \( \alpha \), i.e.

\[
P(\Lambda \leq k \mid H_0 \text{ is true}) = \alpha
\]

First we need to find the form of \( \Lambda \) and further more its distribution.

\[
\Lambda = \frac{\prod_{i=1}^{n} f(x_i; \mu_0)}{\prod_{i=1}^{n} f(x_i; \mu_1)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu_0)^2}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_i-\mu_1)^2}} = e^{-\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu_0)^2} e^{\frac{1}{2} \sum_{i=1}^{n} (x_i - \mu_1)^2} = \exp \left\{ -\frac{1}{2} \left[ \sum_{i=1}^{n} (x_i - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \mu_1)^2 \right] \right\}
\]

Thus, \( \Lambda \leq k \) is equivalent to

\[
-2 \ln k \leq -2 \ln \Lambda = \sum_{i=1}^{n} (x_i - \mu_0)^2 - \sum_{i=1}^{n} (x_i - \mu_1)^2
\]

Define \( k^* = -2 \ln k \)

\[
k^* \leq \sum_{i=1}^{n} (x_i^2 - 2\mu_0 x_i + \mu_0^2) - \sum_{i=1}^{n} (x_i^2 - 2\mu_1 x_i + \mu_1^2) = -2(\mu_0 - \mu_1) \sum_{i=1}^{n} x_i + n(\mu_0 - \mu_1)
\]

Suppose \( \mu_0 > \mu_1 \), then we have critical region

\[
\hat{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \leq \frac{-k^* + n(\mu_0 - \mu_1)}{2n(\mu_0 - \mu_1)} = k^{**}
\]

We actually can find the accurate form of \( k \) or \( k^* \). However, it’s unnecessary, \( \Lambda \leq k \) is equivalent to \( \hat{X} \leq k^{**} \), so we just need to find the value of \( k^{**} \). Recall type I error

\[
P(\hat{X} \leq k^{**} \mid H_0 \text{ is true}) = \alpha
\]
Then we could solve \( k^{**} \) by C.L.T.,

\[
k^{**} = \mu_0 - z_\alpha \frac{1}{\sqrt{n}}
\]

We finally decide to reject \( H_0 \) is \( \bar{X} \leq \mu_0 - z_\alpha \frac{1}{\sqrt{n}} \).

**Problem 2(6.5.5)**

(a) The density function of Bernoulli distribution is

\[
f(k_i; p) = p^{k_i}(1-p)^{1-k_i}, k_i = 0, 1
\]

Then the likelihood function is

\[
L(p) = \prod_{i=1}^{n} f(x_i; p) = p^{\sum_{i=1}^{n} k_i}(1-p)^{n-\sum_{i=1}^{n} k_i} = p^k(1-p)^{n-k}
\]

where \( k = \sum_{i=1}^{n} k_i \), denote the number of success. We know that the MLE of \( p \) is \( \frac{k}{n} \). So the likelihood ratio is

\[
\Lambda = \frac{L(\frac{1}{2})}{L(\frac{k}{n})} = \left(\frac{1}{2}\right)^n \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}
\]

The critical region is

\[
\lambda^* \geq \Lambda = \left(\frac{1}{2}\right)^n \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}
\]

which is equivalent to

\[
\ln \lambda^* \geq n \ln \left(\frac{1}{2}\right) - [k \ln k - k \ln n + (n-k) \ln(n-k) - (n-k) \ln n]
\]

Continue

\[
k \ln k + (n-k) \ln(n-k) \geq n \ln \left(\frac{1}{2}\right) + n \ln n - \ln \lambda^*
\]

Define \( \lambda^{**} = n \ln \left(\frac{1}{2}\right) + n \ln n - \ln \lambda^* \). QED.

(b) Recall Binomial question, do you remember earlier we said that to test \( H_1 : p \neq p_0 \), it’s reasonable that we make the critical region \( \left| \hat{p} - p_0 \right| = \left| \frac{k}{n} - p_0 \right| \geq C \). This question tells us that this critical region can be derived from likelihood ratio test. Let \( f(x) = x \ln(x) + (n-x) \ln(n-x) \), we will find

\[
f'(x) = \ln(x) - \ln(n-x)
\]
$x = \frac{n}{2}$ make $f'$ be 0. And

$$f''(x) = \frac{1}{x} + 1 - n - x = \frac{n}{x(n-x)} > 0$$

So $x = \frac{n}{2}$ is the minimizer of $f(\cdot)$. Also, for any $y = n - x$, which is symmetric to $x$ in $\frac{n}{2}$, we have $f(x) = f(y)$. Actually, here is the graphic of $f(x)$, when $n = 20$

It’s easy to see that if we expect $f(x) \geq \lambda^*$, we need $x$ is away from $\frac{n}{2}$. This implies the critical region is equivalent to

$$\left| k - \frac{n}{2} \right| \geq c$$

where $c$ is some constant. Let $\bar{k} = \frac{k}{n}$. QED

**Problem 3(7.3.5)**

Since we know

$$\frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$$

And if a random variable $X$ follows $\chi^2(n)$, then $E(X) = n$, $\text{Var}(X) = 2n$. Thus

$$E \left( \frac{n-1}{\sigma^2} S^2 \right) = n - 1 \Rightarrow E(S^2) = \sigma^2$$

$$\text{Var} \left( \frac{n-1}{\sigma^2} S^2 \right) = 2(n - 1) \Rightarrow \text{Var}(S^2) = \frac{2\sigma^4}{n-1}$$
By Chebyshev’s Theorem,
\[ P(|S^2 - \sigma^2| < \epsilon) > 1 - \frac{\text{Var}(S^2)}{\epsilon^2} \to 1 \]
So \( S^2 \) is consistent for \( \sigma^2 \).

**Problem 4(7.3.11)**

If random variable \( F \) follows F-distribution with degree \((m, n)\), then there exist \( U \) and \( V \) are independent \( \chi^2 \) random variables with \( m \) and \( n \) degrees of freedom, and
\[ F = \frac{V/m}{U/n} \]
Then \( \frac{1}{F} = \frac{U/n}{V/m} \) has an F-distribution with degree \((n, m)\).

**Problem 5(7.4.11)**

For \( n = 24, t_{\alpha/2,n-1} = t_{0.05,23} = 1.7139 \). For these data \( \bar{y} = 193.54 \) and \( s = \sqrt{\frac{24 \cdot 959265 - 3645^2}{24 \cdot 23}} = 51.19 \). The confidence interval is
\[ \left( 193.54 - 1.7139 \cdot \frac{51.19}{\sqrt{24}}, 193.54 + 1.7139 \cdot \frac{51.19}{\sqrt{24}} \right) = (175.6, 211.4) \]
The medical and statistical definition of ”normal” differ somewhat. There are people with medically norm platelet counts who appear in the population less than 10% of the time.

**Problem 6(7.4.20)**

\( H_0 : \mu = 0.618 \) should be rejected in favor of a two-sided \( H_1 \) at the 0.01 level of significance if \( |t| \geq t_{0.005,33} = 2.75333 \). Given that \( \bar{y} = 0.6373 \) and \( s = 0.14139 \), the \( t \) statistic is
\[ t = \frac{0.6373 - 0.618}{0.14139 \sqrt{34}} = 0/8 \]
So \( H_0 \) is not rejected. These data do not rule out the possibility that national flags embrace the Golden Rectangle as an aesthetic standard.

**Problem 7(7.4.23)**

Because of the skewed shape of \( f_Y(y) \), and if the sample size was small, it would not be unusual for all the \( y_i \)'s to lie close together near 0. When that
happens, \( \bar{y} \) will be less than \( \mu \), \( s \) will be considerably smaller than \( E(S) \), and the \( t \) ratio will be further to the left of 0 than \( f_{r_{n-1}}(t) \) would predict.

Problem 8 (R programming)

```r
> ### Part(a)
> ### Generate one sample
> x=rexp(100,2)
> ### MLE of lambda
> lambda.hat=1/mean(x)
> ### Use if sentence to define when we reject H0
> if(lambda.hat<=2-1.96*sqrt(2)/10||lambda.hat>=2+1.96*sqrt(2)/10){
>   print('Reject H0')
> }else{
>   print('Fail to reject H0')
> }
> [1] "Fail to reject H0"

> ### Part(b)
> ### cov stands for coverage probability, which refers to type I error
> ### in this question.
> ### To do the simulation, our goal is that we use the critical region
> ### found before, to verify whether type I error is or is close to 0.05
> ### First define a vector for coverage probability
> cov=NA
> ### Generate 100 samples, and save the outputs into vector cov
> for (i in 1:100){
>   x=rexp(100,2)
>   lambda.hat=1/mean(x)
>   cov[i]=lambda.hat<=2-1.96*sqrt(2)/10||lambda.hat>=2+1.96*sqrt(2)/10
> }
> ### Type I error
> mean(cov)
> [1] 0.05

> ### Part(c)
> ### To plot curve of power, we need to compute type II error when lambda
> ### is different value.
> ### For each lambda, everything is same as part(b), the only difference is
> ### that we need to use lambda1 instead of
> ### lambda0 (which is 2)
> ### Define a vector to save power
> power=NA
> lambda=seq(0.1,4,by=0.1)
> for (j in 1:length(lambda)){
+ cov=NA
+ for (i in 1:100){
+   x=rexp(100,2)
+   lambda.hat=1/mean(x)
+   cov[i]=lambda.hat<=lambda[j]-1.96*lambda[j]/10||lambda.hat>=lambda[j]+1.96
+   *lambda[j]/10
+ }
+ power[j]=mean(cov)
+ }
> plot(lambda,power,type="l")