Math 109: Winter 2014
Final Exam Practice Questions

Problem 1: Construct a truth table for the statement \((P \text{ and } (P \Rightarrow Q)) \Rightarrow Q\).

Problem 2: Let \(a_1, a_2, \ldots\) be a sequence of real numbers. We say \(\lim_{n \to \infty} a_n = 7\) if for all \(\epsilon > 0\), there exists \(N > 0\) such that \(n > N\) implies \(|a_n - 7| < \epsilon\). What is the negation of \(\lim_{n \to \infty} a_n = 7\)?

Problem 3: Accurately state and prove the division theorem.

Problem 4: Prove or disprove each of the following statements.

   (1) For every integer \(a\), \(3\mid a\) if and only if \(3\mid a^2\).
   (2) For every integer \(a\), \(4\mid a\) if and only if \(4\mid a^2\).

Problem 5: (a) Carefully define what it means for a set to be “denumerable” and (b) prove that \(\mathbb{Z}\) is denumerable using your definition.

Problem 6: Let \(X = \{a, b, c\}\) and \(Y = \{c, d, e\}\). Let \(A = \{a, b\}\). Define \(f : X \to Y\) by \(f(a) = c, f(b) = e, f(c) = e\). Write down the graphs \(G_f\) and \(G_f|_A\).

Problem 7: Let \(S\) denote the set of all infinite ternary sequences, i.e., sequences of the form \((a_1, a_2, a_3, \ldots)\) where \(a_n \in \{0, 1, 2\}\) for all \(n \in \mathbb{Z}^+\). Is \(S\) countable? Prove your claim.

Problem 8: Prove or disprove the following statements.

   (1) For all sets \(X\) and \(Y\) and all functions \(f : X \to Y\), there exists \(A \subseteq X\) such that \(f|_A\) is injective.
   (2) For all sets \(X\) and \(Y\) and all functions \(f : X \to Y\), there exists \(A \subseteq X\) such that \(f|_A\) is surjective.

Problem 9: Let \(X = \{a, b, c, d\}\) and \(Y = \{e, f, g\}\). What are the cardinalities of the following sets?

   (1) \(\{f : X \to Y : f\text{ is a function}\}\).
   (2) \(\{f : Y \to X : f\text{ is a function}\}\).
   (3) \(\{f : X \to Y : f\text{ is a surjection}\}\).
   (4) \(\{f : Y \to X : f\text{ is a surjection}\}\).
   (5) \(\{f : X \to Y : f\text{ is an injection}\}\).
   (6) \(\{f : Y \to X : f\text{ is an injection}\}\).

Problem 10: Which of the following relations on the following sets are equivalence relations? Justify your answers.

   (1) \(\mathbb{Z}^+\) with the relation \(a \sim b\) if and only if \(a/b \geq 1\).
   (2) \(\mathbb{Z}\) with the relation \(a \sim b\) if and only if 4 divides \((a - b)\).
   (3) \(\mathbb{R}\) with the relation \(a \sim b\) if and only if \((a - b) \in \mathbb{Z}\).
   (4) The set of all functions \(\mathbb{R} \to \mathbb{R}\) with \(f \sim g\) if and only if \(f(4) = g(4)\).

Problem 11: Prove that for all \(n \in \mathbb{Z}^+\), we have \(1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}\).