MATH 109A, FINAL

Monday, March 16th, 2015, 11:30–2:29 PM, location TBA

• Your Name: SOLUTIONS.
Problem 1. Let \( \mathbb{N}_7 = \{1, 2, 3, 4, 5, 6, 7\} \).

(a) What is the cardinality of its power set \( \mathcal{P}(\mathbb{N}_7) \)?

(b) How many subsets of \( \mathbb{N}_7 \) have precisely 3 elements?

(c) Find the binomial expansion of the polynomial \((x - 1)^7\).

\[
\begin{align*}
(\text{a}) \quad |\mathcal{P}(\mathbb{N}_7)| &= 2^7 = 128 \\
(\text{b}) \quad \binom{7}{3} &= \frac{7!}{3!4!} \quad = \frac{1}{3!} 7 \cdot 6 \cdot 5 = 35 \\
(\text{c}) \quad \text{Pascal's Triangle:} \\
\begin{array}{cccccc}
1 & & & & & \\
1 & 2 & 1 & & & \\
1 & 3 & 3 & 1 & & \\
1 & 4 & 6 & 4 & 1 & \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
\end{array} \\
\Rightarrow \text{Binomial expansion:} \\
(x-1)^7 &= \sum_{k=0}^{7} \binom{7}{k} (-1)^k x^{7-k} \\
&= x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x - 1.
\end{align*}
\]
Problem 2.

(a) Compute the greatest common divisor $\text{GCD}(2015, 273)$. 

\[
\text{GCD}(2015, 273) = 13.
\]

(b) Find two integers $x$ and $y$ such that

\[
\text{GCD}(2015, 273) = 2015x + 273y.
\]

(c) What is the (multiplicative) inverse of 21 modulo 155?

\[\text{(c) From (b), by dividing by 13,} \]
\[1 = 8 \cdot 155 + (-59) \cdot 21.\]
\[\Downarrow \]
\[(-59) \cdot 21 \equiv 1 \pmod{155}. \]

So the inverse of 21 modulo 155 is

\[
[21]^{-1} = [-59] = [96].
\]
Problem 3.

(a) Find an explicit formula for $2 + 4 + 6 + \cdots + 2n$, and prove it by induction (or otherwise).

(b) Find an explicit formula for $1 + 3 + 5 + \cdots + (2n - 1)$, and prove it by induction (or otherwise).

(c) Use (a) and (b) to give a formula for the alternating sum $\sum_{k=1}^{n} (-1)^{k-1}k$.

(a) Recall, $T_n = \frac{1}{2}n(n+1) = 1 + 2 + \cdots + n$ ("triangular numbers")

$$2 + 4 + \cdots + 2n = 2(1 + 2 + \cdots + n) = 2T_n = n(n+1).$$

(b) $1 + 3 + \cdots + (2n-1) = \sum_{k=1}^{2n-1} k - \sum_{k=1}^{n} 2k = \frac{T_{2n}}{2} - 2T_n$

$$= \frac{1}{2} \cdot 2n(2n+1) - n(n+1) = 2n^2 + n - n^2 - n = n^2$$

(c) $\sum_{k=1}^{n} (-1)^{k-1}k$ ?

- First, if $n = 2m$ is even,

$$A_{2m} = \left(1 + 3 + \cdots + (2m-1)\right) - \left(2 + 4 + \cdots + 2m\right)$$

$$= m^2 - m(m+1) = -m.$$  

- Second, if $n = 2m - 1$ is odd,

$$A_{2m-1} = \left(1 + 3 + \cdots + (2m-1)\right) - \left(2 + 4 + \cdots + 2(m-1)\right)$$

$$= m^2 - (m-1)m = +m.$$  

Combined,

$$A_n = \sum_{k=1}^{n} (-1)^{k-1}k = (-1)^{n-1} \cdot \left\lfloor \frac{n+1}{2} \right\rfloor.$$
Problem 4. Define a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by \( f(x, y) = (x^2 + y, x - y^2) \).

(a) Draw a sketch of \( f(X) \subset \mathbb{R}^2 \), where \( X = \{(x, 0) : x \in \mathbb{R}\} \) is the \( x \)-axis.

(b) Draw a sketch of \( f(Y) \subset \mathbb{R}^2 \), where \( Y = \{(0, y) : y \in \mathbb{R}\} \) is the \( y \)-axis.

(c) Find the inverse image of the origin, \( f^{-1}(0, 0) \). Is \( f \) injective?

\[ f(x, 0) = (x^2, x) = (t, \pm \sqrt{t}) \quad \text{and} \quad f(0, y) = (y, -y^2) \]

\[ f^{-1}(0, 0) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = (0, 0)\} \]

\[ \star \quad x^2 + y = 0 = x - y^2 \quad \Rightarrow \quad x = y^2 = (-x^2)^2 = x^4 \]

\[ \Rightarrow \quad x = 0 \lor x^3 = 1 \]

\[ \Rightarrow \quad x = 0 \lor x = 1. \]

Get candidates:

\((0, 0), (1, -1)\)

Conversely, they do satisfy \( \star \). So, \( f^{-1}(0, 0) = \{(0, 0), (1, -1)\} \).

- Since \((0, 0)\) has \( \geq 1 \) pre-image, \( f \) is \underline{not} injective.
Problem 5. Let \( L \subseteq \mathbb{R}^2 \) be a line through the origin. Define a relation \( \sim \) on \( \mathbb{R}^2 \) by declaring that
\[
x \sim y \iff x - y \in L.
\]
Here \( x \) and \( y \) are vectors in \( \mathbb{R}^2 \).

(a) Show that \( \sim \) is an equivalence relation on \( \mathbb{R}^2 \).

(b) What is the equivalence class of the origin, \([0]\)?

(c) Give a geometric description of the equivalence class \([x]\) of any \( x \in \mathbb{R}^2 \).

\begin{itemize}
  \item [(i)] \text{REFLEXIVE: } x \sim x \iff 0 = x - x \in L \quad \text{OK, } L \subseteq \mathbb{R}^2 \text{ is a subspace.}
  \item [(ii)] \text{SYMMETRIC: } x \sim y \iff x - y \in L
  \quad \iff y - x \in L
  \quad \iff y \sim x
  \text{ (closed under scaling)}
  \text{Here, by -1.}
  \item [(iii)] \text{TRANSITIVE: } x \sim y \land y \sim z \iff x - y \in L \land y - z \in L
  \implies (x - y) + (y - z) \in L
  \iff x \sim z.
\end{itemize}

(b) \([0] = \{ x \in \mathbb{R}^2 : x - 0 \in L \} = \{ x : x - 0 \in L \} = L \).

(c) \([x] = \{ y \in \mathbb{R}^2 : y \sim x \} = \{ y : y - x \in L \} = x + L \).

\( [x] = x + L \)

\( L = [0] \)

- the line through \( x \), parallel to \( L \).
- \( \sim \) a translate of \( L \) (by the "offset" \( x \)).
Problem 6. For each \( n \in \mathbb{N} \), consider the following subset of the plane,

\[
A_n = \{(x, y) \in \mathbb{R}^2 : \frac{1}{n} < x^2 + y^2 < n\}.
\]

(a) What is \( A_1 \)? Draw a sketch of \( A_4 \).

(b) Is \( A_{2014} \subseteq A_{2016} \)?

(c) Find the intersection \( \bigcap_{n=5}^{\infty} A_n \).

(d) Find the union \( \bigcup_{n=5}^{\infty} A_n \).

(a) \( A_1 = \emptyset \quad (1 < x^2+y^2 < 1) \),

(b) In general, \( A_n \subseteq A_{n+1} \): \( \frac{1}{n+1} < \frac{1}{n} < x^2+y^2 < n < n+1 \).

(c) \( A_5 \subseteq A_6 \subseteq A_7 \subseteq \ldots \), \( \bigcap_{n=5}^{\infty} A_n = A_5 \).

(d) \( \bigcup_{n=5}^{\infty} \mathbb{R}^2 \setminus \{(0,0)\} \).

\( \subseteq: \frac{1}{n} < x^2 + y^2 < n \Rightarrow (x,y) \neq (0,0) \).

\( \supseteq: \) If \( (x,y) \neq (0,0) \), \( x^2 + y^2 > 0 \).

May find \( N \) s.t. \( x^2 + y^2 > \frac{1}{N} \).

By choosing \( N \) even larger, can also arrange that \( N > x^2 + y^2 \). Then \( (x,y) \in A_N \). \( \square \)
Problem 7. For each of the functions defined below, indicate whether or not they are injective, surjective, and/or bijective. Justify your answer.

(a) (Injective, Surjective, Bijective)

\[ f : \mathbb{N} \to \mathbb{N}, \quad f(n) = n^2. \]

(b) (Injective, Surjective, Bijective)

\[ f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x, y) = x + y^2. \]

(c) (Injective, Surjective, Bijective)

\[ f : \mathbb{Q} \to \mathbb{R}, \quad f(x) = x^4 + \sqrt{2}. \]

(d) (Injective, Surjective, Bijective)

\[ f : \mathbb{Z} \to \mathbb{Z}, \quad f(n) = 2015 - n. \]

(a) Injective: \[ m^2 = n^2 \implies m = \pm n \implies m = n. \]

Not Surjective: \[ 2 \neq n^2, \forall n. \quad \text{since} \ m, n > 0. \]

(b) Not injective: \[ f(0,1) = 1 = f(0,-1) \]

Surjective: \[ \forall x \in \mathbb{R}, x = f(x,0). \]

(c) Not injective: \[ f(1) = 1 + \sqrt{2} = f(-1). \]

Not surjective: \[ f(x) \geq \sqrt{2}, \text{so } 0 \text{ not in } \text{Im}(f). \]

(d) Bijective (i.e., injective & surjective): \[ 2015 - n = m \]

\[ f(f(m)) = 2015 - (2015 - n) = n. \]

\[ f \text{ is its own inverse function: } f = f^{-1}. \]
Problem 8. Which of the following sets are countable? Explain.

C. (a) \( \mathbb{Q} \times \mathbb{Z} \times \mathbb{N} \).

V. (b) \( \mathcal{P}(\mathbb{Z}) \).

V. (c) The set of all functions \( f : \mathbb{Q} \to \mathbb{Q} \).

C. (d) The set of all functions \( f : \mathbb{N} \to \mathbb{N} \).

(a) \( \mathbb{Q} \) is countable \((14.2.6)\), then so is \( \mathbb{Z} \subset \mathbb{Q} \) and \( \mathbb{N} \) \((14.2.5)\). Therefore \( \mathbb{Q} \times \mathbb{Z} \times \mathbb{N} \) is countable \((14.2.3)\).

(b) \( \mathcal{P}(\mathbb{Z}) \) is equipotent to the set of all functions \( f : \mathbb{Z} \to \{0,1\}^\mathbb{Z} \). (take \( A \) to its char. function \( \chi_A \)). The latter is uncountable by Cantor’s diagonal argument: Suppose we can list all such functions, \( \{f_1, f_2, \ldots\} \). Define \( f : \mathbb{Z} \to \{0,1\}^\mathbb{Z} \) by

\[
\begin{align*}
  f(n) &\df . \begin{cases} 1, & f_n(n) = 0, \\ 0, & f_n(n) = 1 \end{cases} \\
  (n &> 1) \end{align*}
\]

Then \( f \not= f_n, \forall n \), so this \( f \) is not on the list.

(c) This set contains the \( \{0,1\} \)-valued functions on \( \mathbb{Q} \), which is uncountable by Cantor’s argument; as in (b).

(d) This set is equipotent to \( \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N} \), which is countable by taking \( f : \mathbb{N} \to \mathbb{N} \) to the tuple \( (f(1), \ldots, f(\alpha)) \).
Problem 9. Let \( x \in \mathbb{R} \) be an irrational number, and let \( N \in \mathbb{N} \) be a fixed positive integer. Show that there exists a rational number \( \frac{a}{b} \) satisfying the following two conditions:

\[
\begin{align*}
&1 \leq b \leq N, \\
&|x - \frac{a}{b}| \leq \frac{1}{bN}.
\end{align*}
\]

Hint: Divide \([0, 1] \) into \( N \) holes, and consider \( N+1 \) pigeons \( 0, \langle x \rangle, \langle 2x \rangle, \ldots, \langle Nx \rangle \), where \( \langle x \rangle = x - [x] \) denotes the fractional\(^1\) part of \( x \).

\[
\text{Divide } [0, 1] \text{ into } N \text{ "holes": } [0, \frac{1}{N}) \cup \left[ \frac{1}{N}, \frac{2}{N} \right) \cup \ldots \cup \left[ \frac{N-1}{N}, 1 \right) \text{.}
\]

Observe that \( 0, \langle x \rangle, \langle 2x \rangle, \ldots, \langle Nx \rangle \) lie in \([0, 1)\), and they’re distinct — since \( x \) is irrational. \( \langle ix \rangle = \langle jx \rangle \) then \( (i-j)x = \lfloor ix \rfloor - \lfloor jx \rfloor \in \mathbb{Z} \).

By the PIGEONHOLE PRINCIPLE, \( \exists i \neq j \) s.t. \( \langle ix \rangle \) and \( \langle jx \rangle \) lie in the same hole.

Assume \( i > j \), without loss of generality.

\[
\begin{align*}
-\frac{1}{N} &< \langle ix \rangle - \langle jx \rangle < \frac{1}{N} \\
\end{align*}
\]

Then \( \left| b(x-a) \right| < \frac{1}{N} \), and \( 1 \leq b \leq N \). Done. \( \square \)

\(^1\)The \textit{Floor} \([x]\) is the largest integer \( \leq x \).