Problem 1. Prove that the cube root of 12 is an irrational number.

Proof. Suppose that $12^{1/3}$ is rational, i.e. there exist relatively prime integers $a$ and $b$ such that

$$12^{1/3} = \frac{a}{b}.$$ 

Then

$$a^3 = 12b^3 = 2^33b^3.$$ 

So 3 divides $a^3$. Since 3 is prime, 3 divides $a$ and so $3^3$ divides $a^3$. Then $3^3$ divides $2^33b^3$. Since 3 is prime, 3 divides $b$ contradicting that $a$ and $b$ are relatively prime. □

Problem 2. Describe an explicit method for constructing a bijection between the set of rational numbers and the set of positive integers.

Proof. The key here is to define some function $f : \mathbb{N} \to \mathbb{Q}$ that hits every element of $\mathbb{Q}$ exactly once. We construct a diagram similar to the one on page 29 of Rudin in the proof of theorem 2.12.

Then we define our map to go diagonally as in Rudin’s diagram, skipping repeated elements of $\mathbb{Q}$. Then we have $f(1) = 0$, $f(2) = -1$, $f(3) = -1/2$, $f(4) = 1$, $f(5) = -1/3$, . . . . This gives a bijection from $\mathbb{N} \to \mathbb{Q}$. □

Problem 3. Jane claims that she has found a pair of real numbers $a < b$ such that the interval $(a, b) \subset \mathbb{R}$ contains no irrational numbers. Prove that Jane is mistaken.

Proof. Suppose there exist a pair of real numbers $a < b$ such that the interval $(a, b)$ contains no irrational numbers. Then $(a, b) \subset \mathbb{Q}$ and hence is countable. This is a contradiction since any interval in $\mathbb{R}$ is uncountable. □

Problem 4. Let $L$ denote the $x$-axis in the usual Cartesian plane $\mathbb{R}^2$. Give an example of a closed set $E$ in the plane which has points arbitrarily close to $L$, but such that $E$ is disjoint from $L$. Does such an example exist if $L$ were the circle $x^2 + y^2 = 1$ instead of the $x$-axis?

Proof. The graph of the function $f(x) = 1/x$ on the interval $(0, \infty)$ does the trick, i.e. let

$$E = \left\{ \left( x, \frac{1}{x} \right) : x > 0 \right\}.$$ 

If $L$ were the circle $x^2 + y^2 = 1$, no such example exists. The reason here is because the unit circle is a compact subset of $\mathbb{R}^2$, and then the following theorem applies:
Theorem 1. Let $X$ be a metric space. $L \subseteq X$ be compact and let $E \subseteq X$ be closed. Then $d(L, E) > 0$.  

Problem 5. Let $E$ be the set of those real numbers in the interval $(0,1)$ with infinite decimal expansions $p_1p_2p_3\ldots$ such that at least one of the digits $p_i$ is 0 or 9. Is $E$ open in $\mathbb{R}$? Justify.

Proof. Yes, $E$ is open in $\mathbb{R}$. To do this we show that if $p \in E$ then there is a neighborhood of $p$ contained in $E$.

Let $p = 0.p_1p_2p_3\ldots \in E$. Then there exists an integer $k$ such that $p_k$ is 0 or 9. Let $r = 10^{-(k+1)}$. Now we have to consider several cases.

If $p_{k+1}$ is not 0 or 9, it is clear that $N_r(p) \subseteq E$ since every element of $N_r(p)$ has the $k$th digit equal to $p_k$.

If $p_k = 0$ and $p_{k+1} = 0$, then any element in $N_r(p)$ has the $k$th digit equal to 0 or 9.

If $p_k = 0$ and $p_{k+1} = 9$, then any element in $N_r(p)$ has either the $k$th digit equal to 0 or the $(k+1)$th digit equal to 9.

If $p_k = 9$ and $p_{k+1} = 0$, then any element in $N_r(p)$ has either the $k$th digit equal to 9 or the $(k+1)$th digit equal to 0.

If $p_k = 9$ and $p_{k+1} = 9$, then any element in $N_r(p)$ has the $k$th digit equal to 0 or 9.

In any of the above cases, $N_r(p) \subseteq E$ and hence $E$ is open.

Problem 6. Let $E$ be a bounded open subset of $\mathbb{R}$ such that $0 \in E$. Let $M = \{x \in E : [0, x] \subseteq E\}$. Let $\alpha$ denote the least upper bound of $M$ in $\mathbb{R}$. Prove that $\alpha \notin M$.

Proof. Suppose $\alpha \in M$. Then by definition of $M$, $[0, \alpha] \subseteq E$, in particular, $\alpha \in E$. Since $E$ is open, there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subseteq E$. Then we have that $[0, \alpha + r/2] \subseteq [0, \alpha + r] = [0, \alpha] \cup (\alpha - r, \alpha + r) \subseteq E$.

Then by definition of $M$, $\alpha + r/2 \in M$ and $\alpha + r/2 > \alpha$, contradicting that $\alpha = \sup M$.

Hence $\alpha \notin M$ by contradiction.