1.8 Riesz Representation Theorem

In these notes there will be two primary sources of measures to which we will apply the foregoing abstract theory: these are (a) Hausdorff measures, constructed in Chapter 2, and (b) Radon measures characterizing certain linear functionals. Let $\mathcal{L}^n : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying

$$\sup \{ |\mathcal{L}(f)| : f \in C_c(\mathbb{R}^n, \mathbb{R}^m), |f| \leq 1, \text{ supp}(f) \subseteq K \} < \infty$$

for each compact set $K \subseteq \mathbb{R}^n$. Then there exists a Radon measure $\mu$ on $\mathbb{R}^n$ and a $\mu$-measurable function $s : \mathbb{R}^n \to \mathbb{R}^m$ such that

1. $|s(x)| = 1$ for $\mu$-a.e. $x$, and
2. $\mathcal{L}(f) = \int_{\mathbb{R}^n} f \cdot s \, d\nu$,

for all $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$.

DEFINITION We call $\mu$ the variation measure, defined for each open set $V \subseteq \mathbb{R}^n$ by

$$\mu(V) \equiv \sup \{ |\mathcal{L}(f)| : f \in C_c(\mathbb{R}^n, \mathbb{R}^m), |f| \leq 1, \text{ supp}(f) \subseteq V \}.$$

PROOF

1. Define $\mu$ on open sets $V$ as above and then set

$$\mu(A) = \inf \{ \mu(V) : A \subseteq V \}$$

for arbitrary $A \subseteq \mathbb{R}^n$.

2. Claim (a): $\mu$ is a measure.

Proof of Claim (a): Let $V_j, \{V_j\}_{j=1}^\infty$ be open subsets of $\mathbb{R}^n$, with $V \subseteq \bigcup_{j=1}^\infty V_j$. Choose $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ such that $g \leq 1$ and $\text{ supp}(g) \subseteq V$. Since $\text{ supp}(g)$ is compact, there exists an index $k$ such that $\text{ supp}(g) \subseteq \bigcup_{j=1}^k V_j$. Let $\{A_j\}_{j=1}^\infty$ be a finite sequence of smooth functions such that $\text{ supp}(A_j) \subseteq V_j$ for $1 \leq j \leq k$ and $\sum_{j=1}^\infty A_j = 1$ on $\text{ supp}(g)$. Then $g = \sum_{j=1}^\infty A_j$, and so

$$|\mathcal{L}(g)| = \sum_{j=1}^\infty |\mathcal{L}(A_j)| \leq \sum_{j=1}^k |L(g_j)| \leq \sum_{j=1}^\infty \mu(V_j).$$

Then, taking the supremum over $g$, we find $\mu(V) \leq \sum_{j=1}^\infty \mu(V_j)$.

Next let $\{A_j\}_{j=1}^\infty$ be arbitrary sets with $A \subseteq \bigcup_{j=1}^\infty A_j$. Fix $e > 0$. Choose open sets $V_j$...
such that \( A_j \subset V_j \) and \( \mu(A_j) + \epsilon/2^j \geq \mu(V_j) \). Then

\[
\mu(A) \leq \sum_{j=1}^{\infty} \mu(V_j) \leq \sum_{j=1}^{\infty} \mu(A_j) + \epsilon.
\]

3. Claim #3: \( \mu \) is a Radon measure.

Proof of Claim #3: Let \( U_1 \) and \( U_2 \) be open sets with \( \text{dist}(U_1, U_2) > 0 \). Then \( \mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2) \) by definition of \( \mu \). Hence if \( A_1, A_2 \subset \mathbb{R}^n \) and \( \text{dist}(A_1, A_2) > 0 \), then \( \mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) \). According to Caratheodory's Criterion (Section 1.1.1), \( \mu \) is a Borel measure. Furthermore, by its definition, \( \mu \) is Borel regular; indeed, given \( A \subset \mathbb{R}^n \), there exist open sets \( V_0 \) and \( V_1 \) such that \( A \subset V_0 \) and \( \mu(V_1) \leq \mu(A) + \epsilon \) for all \( \epsilon > 0 \). Thus \( \mu(A) = \mu(V_0 \cap V_1) \).

Finally, condition (i) implies \( \mu(N) < \infty \) for all compact set \( N \).

4. Now, let \( C_0^\infty(\mathbb{R}^n) = \{ f \in C_c(\mathbb{R}^n) \mid f \geq 0 \} \), and for \( f \in C_0^\infty(\mathbb{R}^n) \), set

\[
\lambda(f) = \sup \{ |f(g)| \mid g \in C_c(\mathbb{R}^n), |g| \leq f \}
\]

Observe that for all \( f_1, f_2 \in C_0^\infty(\mathbb{R}^n) \), \( f_1 \leq f_2 \) implies \( \lambda(f_1) \leq \lambda(f_2) \). Also \( \lambda(cf) = c\lambda(f) \) for all \( c \geq 0 \). \( f \in C_0^\infty(\mathbb{R}^n) \).

5. Claim #5: For all \( f_1, f_2 \in C_0^\infty(\mathbb{R}^n) \), \( \lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2) \).

Proof of Claim #5: If \( g_1, g_2 \in C_c(\mathbb{R}^n) \) with \( |g_1| \leq f_1 \) and \( |g_2| \leq f_2 \), then \( |g_1 + g_2| \leq f_1 + f_2 \). We can furthermore assume \( L(g_1), L(g_2) \geq 0 \). Therefore,

\[
\lambda(g_1) + \lambda(g_2) = L(g_1 + g_2) \leq \lambda(f_1 + f_2).
\]

Taking suprema over \( g_1 \) and \( g_2 \) with \( g_1, g_2 \in C_c(\mathbb{R}^n) \) gives

\[
\lambda(f_1 + f_2) \geq \lambda(f_1) + \lambda(f_2).
\]

Now fix \( g \in C_c(\mathbb{R}^n) \), with \( |g| \leq f_1 + f_2 \). Set

\[
g_t = \begin{cases} 
\frac{g}{f_1 + f_2} & \text{if } f_1 + f_2 > 0, \\
0 & \text{if } f_1 + f_2 = 0.
\end{cases}
\]

for \( t = 1, 2 \). Then \( g_1, g_2 \in C_c(\mathbb{R}^n) \) and \( g = g_1 + g_2 \). Moreover, \( |g_t| \leq f_t \), \( (t = 1, 2) \), so that

\[
\lambda(g) \geq \lambda(g_1) + \lambda(g_2) \geq \lambda(f_1) + \lambda(f_2).
\]

Consequently,

\[
\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2).
\]
Hence
\begin{align*}
\lambda(f) &= \lambda \left( f - f \sum_{j=1}^{N} h_j \right) + \lambda \left( f \sum_{j=1}^{N} h_j \right) \\
&\leq \varepsilon \|f\|_{L^\infty} + \sum_{j=1}^{N} \lambda(fh_j) \\
&\leq \varepsilon \|f\|_{L^\infty} + \sum_{j=1}^{N} t_j \mu(U_j)
\end{align*}
and
\begin{align*}
\lambda(f) &\geq \sum_{j=1}^{N} \lambda(fh_j) \\
&\geq \sum_{j=1}^{N} t_j \mu(U_j) - t_N \varepsilon.
\end{align*}

Finally, since
\[ \sum_{j=1}^{N} t_j \mu(U_j) < \int_{\mathbb{R}^n} f \, d\mu < \sum_{j=1}^{N} t_j \mu(U_j), \]
we have
\[ |\lambda(f)| \leq \int_{\mathbb{R}^n} |f| \, d\mu < \varepsilon + \int_{\mathbb{R}^n} |f| \, d\mu = \varepsilon + \varepsilon = 2\varepsilon. \]

7. Claim #5: There exists a $\mu$-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ satisfying (6).

Proof of Claim #5: Fix $\varepsilon \in \mathbb{R}^m$, $|\varepsilon| = 1$. Define $\lambda_\varepsilon(f) \equiv L(\varepsilon f)$ for $f \in C_0(\mathbb{R}^n)$. Then $\lambda_\varepsilon$ is linear and
\[ |\lambda_\varepsilon(f)| = |L(\varepsilon f)| \\
\leq \sup \{ |L(g)| : g \in C_0(\mathbb{R}^n, \mathbb{R}^m), |g| \leq |f| \} \\
= \lambda(|f|) = \int_{\mathbb{R}^n} |f| \, d\mu. \]

thus we can extend $\lambda_\varepsilon$ to a bounded linear functional on $L^1(\mathbb{R}^n; \mu)$. Hence there exists $\sigma_\varepsilon \in L^\infty(\mu)$ such that
\[ \lambda_\varepsilon(f) = \int_{\mathbb{R}^n} f \sigma_\varepsilon \, d\mu \quad (f \in C_0(\mathbb{R}^n)). \]

Let $\varepsilon_1, \ldots, \varepsilon_m$ be the standard basis for $\mathbb{R}^m$ and define $\nu = \sum_{j=1}^{m} \varepsilon_j \varepsilon_j^T$. Then if $f \in C_0(\mathbb{R}^n; \mathbb{R}^m)$, we have
\begin{align*}
L(f) &= \sum_{j=1}^{m} L(f \cdot \varepsilon_j) \\
&= \sum_{j=1}^{m} \int_{\mathbb{R}^n} (f \cdot \varepsilon_j)^2 \tau_\varepsilon \, d\mu \\
&= \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu.
\end{align*}

8. Claim #6: $|\sigma| = 1 \mu$ a.e.

Proof of Claim #6: Let $U \subset \mathbb{R}^n$ be open, $\mu(U) < \infty$. By definition,
\[ \mu(U) = \sup \left\{ \int_{\mathbb{R}^n} f \, d\nu : f \in C_0(\mathbb{R}^n; \mathbb{R}^m), |f| \leq 1, \text{ spt} \, (f) \subset U \right\}. \]

Now take $f_k \in C_0(\mathbb{R}^n; \mathbb{R}^m)$ such that $|f_k| \leq 1$, spt $(f_k) \subset U$, and $f_k \to \sigma$ in $\mu$ a.e.; such functions exist by Corollary 1 in Section 1.2. Then
\[ \int_{\mathbb{R}^n} |\sigma| \, d\mu = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k \cdot \sigma \, d\mu = \mu(U) \]
by (**) if $f_k \cdot \sigma \leq \mu(U)$ for all open $U \subset \mathbb{R}^n$; hence $|\sigma| = 1$ $\mu$ a.e.

On the other hand, if $f \in C_0(\mathbb{R}^n; \mathbb{R}^m)$ with $|f| \leq 1$ and spt $(f) \subset U$, then
\[ \int_{\mathbb{R}^n} f \cdot \sigma \, d\mu \leq \int_{\mathbb{R}^n} |\sigma| \, d\mu. \]

Consequently (**) implies
\[ \mu(U) \leq \int_{\mathbb{R}^n} |\sigma| \, d\mu. \]

Thus $\mu(U) = \int_{\mathbb{R}^n} |\sigma| \, d\mu$ for all open $U \subset \mathbb{R}^n$; hence $|\sigma| = 1$ $\mu$ a.e.

An immediate and very useful application is the following characterization of nonnegative linear functionals.

**Corollary 1**

Assume $L : C_0(\mathbb{R}^n) \to \mathbb{R}$ is linear and nonnegative, so that
\[ L(f) \geq 0 \quad \text{for all} \ f \in C_0(\mathbb{R}^n), f \geq 0. \]

(g)
Then there exists a Radon measure $\mu$ on $\mathbb{R}^n$ such that

$$L(f) = \int_{\mathbb{R}^n} f \, d\nu$$

for all $f \in C_c^0(\mathbb{R}^n)$.

**Proof** Choose any compact set $K \subset \mathbb{R}^n$, and select a smooth function $\zeta$ such that $\xi$ has compact support, $\xi \equiv 1$ on $K$, $0 \leq \zeta \leq 1$. Then for any $f \in C_c^0(\mathbb{R}^n)$ with sup $\{f\} < K$, set $g = \|f\|_L^\infty \zeta - f \geq 0$. Therefore (a) implies

$$0 \leq L(g) = \|f\|_L^\infty L(\zeta) - L(f),$$

and so

$$L(f) < C \|f\|_L^\infty$$

for $C = L(\zeta)$. $L$ thus extends to a linear mapping from $C_c^0(\mathbb{R}^n)$ into $\mathbb{R}$, satisfying the hypothesis of the Riesz Representation Theorem. Hence there exist $\mu, \sigma$ as above so that

$$L(f) = \int_{\mathbb{R}^n} f \, d\mu \quad (f \in C_c^0(\mathbb{R}^n))$$

with $\sigma = -1 \mu$ a.e. But then (c) implies $\sigma = \sigma$ a.e.  

1.9 Weak convergence and compactness for Radon measures

We introduce next a notion of weak convergence for measures.

**Theorem 1**

Let $\mu_k$ be Radon measures on $\mathbb{R}^n$. The following three statements are equivalent:

(i) $\lim_{k \to \infty} \int_{\mathbb{R}^n} f \, d\mu_k = \int_{\mathbb{R}^n} f \, d\nu$ for all $f \in C_c(\mathbb{R}^n)$

(ii) $\lim_{k \to \infty} \mu_k(K) \leq \mu(K)$ for each compact set $K \subset \mathbb{R}^n$ and $\mu(U) \leq \liminf_{k \to \infty} \mu_k(U)$ for each open set $U \subset \mathbb{R}^n$.

(iii) $\lim_{k \to \infty} \mu_k(B) = \mu(B)$ for each bounded borel set $B \subset \mathbb{R}^n$ with $\mu(B) = 0$.

**Definition** If (i) through (iii) hold, we say the measures $\mu_k$ converge weakly to the measure $\mu$, written

$$\mu_k \rightharpoonup \mu.$$