Minkowski’s inequality for integrals

The following inequality is a generalization of Minkowski’s inequality C12.4 to double integrals. In some sense it is also a theorem on the change of the order of iterated integrals, but equality is only obtained if \( p = 1 \).

13.14 Theorem (Minkowski’s inequality for integrals) Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be \(\sigma\)-finite measure spaces and \(u : X \times Y \to \mathbb{R}\) be \(\mathcal{A} \otimes \mathcal{B}\)-measurable. Then

\[
\left( \int_X \left( \int_Y |u(x, y)| \, \nu(dy) \right)^p \mu(dx) \right)^{1/p} \leq \int_Y \left( \int_X |u(x, y)|^p \, \mu(dx) \right)^{1/p} \, \nu(dy)
\]

holds for all \( p \in [1, \infty) \), with equality for \( p = 1 \).

Proof If \( p = 1 \), the assertion follows directly from Tonelli’s theorem 13.8. If \( p > 1 \) we set

\[
U_k(x) := \left( \int_Y |u(x, y)| \, \nu(dy) \wedge k \right) \mathbb{1}_{A_k}(x)
\]

where \( A_k \in \mathcal{A} \) is a sequence with \( A_k \uparrow X \) and \( \mu(A_k) < \infty \). Without loss of generality we may assume that \( U_k(x) > 0 \) on a set of positive \( \mu \)-measure, otherwise the left-hand side of the above inequality would be 0 (using Beppo Levi’s theorem 9.6) and there would be nothing to prove. By Tonelli’s theorem and Hölder’s inequality T12.2 with \( \frac{1}{p} + \frac{1}{q} = 1 \) or \( q = \frac{p}{p-1} \), we find

\[
\int_X U_k^p(x) \, \mu(dx) \leq \int_X U_k^{p-1}(x) \left( \int_Y |u(x, y)| \, \nu(dy) \right) \mu(dx)
\]

\[
= \int_Y \int_X U_k^{p-1}(x) |u(x, y)| \, \mu(dx) \, \nu(dy)
\]

\[
\leq \int_Y \left( \int_X U_k^p(x) \, \mu(dx) \right)^{1-1/p} \left( \int_X |u(x, y)|^p \, \mu(dx) \right)^{1/p} \nu(dy).
\]

The claim follows upon dividing both sides by \( \left( \int_X U_k^p(x) \, \mu(dx) \right)^{1-1/p} \) and letting \( k \to \infty \) with Beppo Levi’s theorem 9.6.

Problems

13.1. Prove the rules (13.2) for Cartesian products.

13.2. Let \((X, \mathcal{A}, \mu)\) and \((Y, \mathcal{B}, \nu)\) be two \(\sigma\)-finite measure spaces. Show that \(A \times N\), where \( A \in \mathcal{A} \) and \( N \in \mathcal{B} \), \( \nu(N) = 0 \), is a \( \mu \times \nu \)-null set.
13.3. Denote by \( \lambda \) Lebesgue measure on \((0, \infty)\). Prove that the following iterated integrals exist and that
\[
\int_{(0, \infty)} \int_{(0, \infty)} e^{-xy} \sin x \sin y \lambda(dx)\lambda(dy) = \int_{(0, \infty)} \int_{(0, \infty)} e^{-xy} \sin x \sin y \lambda(dy)\lambda(dx).
\]

Does this imply that the double integral exists?

13.4. Denote by \( \lambda \) Lebesgue measure on \((0, 1)\). Show that the following iterated integrals exist, but yield different values:
\[
\int_{(0, 1)} \int_{(0, 1)} \frac{x^2 - y^2}{(x^2 + y^2)^2} \lambda(dx)\lambda(dy) \neq \int_{(0, 1)} \int_{(0, 1)} \frac{x^2 - y^2}{(x^2 + y^2)^2} \lambda(dy)\lambda(dx).
\]

What does this tell about the double integral?

13.5. Denote by \( \lambda \) Lebesgue measure on \((-1, 1)\). Show that the iterated integrals exist, coincide,
\[
\int_{(-1, 1)} \int_{(-1, 1)} \frac{xy}{(x^2 + y^2)^2} \lambda(dx)\lambda(dy) = \int_{(-1, 1)} \int_{(-1, 1)} \frac{xy}{(x^2 + y^2)^2} \lambda(dy)\lambda(dx)
\]
but that the double integral does not exist.

13.6. (i) Prove that \( \int_{(0, \infty)} e^{-tx} \lambda(dt) = \frac{1}{x} \) for all \( x > 0 \).
(ii) Use (i) and Fubini’s theorem to show that the sine integral
\[
\lim_{n \to \infty} \int_{(0, n)} \frac{\sin x}{x} \lambda(dx) = \frac{\pi}{2}.
\]

13.7. Let \( \mu(A) := \#A \) be the counting measure and \( \lambda \) be Lebesgue measure on the measurable space \(([0, 1], \mathcal{B}[0, 1])\). Denote by \( \Delta := \{(x, y) \in [0, 1]^2 : x = y\} \) the diagonal in \([0, 1]^2\). Check that
\[
\int_{[0, 1]} \int_{[0, 1]} 1_\Delta(x, y) \lambda(dx)\mu(dy) \neq \int_{[0, 1]} \int_{[0, 1]} 1_\Delta(x, y) \mu(dy)\lambda(dx).
\]

Does this contradict Tonelli’s theorem?

13.8. (i) State Tonelli’s and Fubini’s theorems for spaces of sequences, i.e. for the measure space \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)\) where \( \mu := \sum_{j \in \mathbb{N}} \delta_j \), and obtain criteria when one can interchange two infinite summations.
(ii) Using similar considerations as in part (i) deduce the following.

Lemma Let \( \{A_j\}_j \) be countably many (i.e. a finite or countably infinite number of) mutually disjoint sets whose union is \( \mathbb{N} \), and let \( \{x_k\}_{k \in \mathbb{N}} \subset \mathbb{R} \) be a sequence. Then
\[
\sum_{k \in \mathbb{N}} x_k = \sum_{j} \sum_{k \in A_j} x_k
\]
in the sense that if either side converges absolutely, so does the other, in which case both sides are equal.

13.9. Let \( u : \mathbb{R}^2 \to [0, \infty] \) be a Borel measurable function. Denote by \( S[u] := \{(x, y) : 0 \leq y \leq u(x)\} \) the set above the abscissa and below the graph \( \Gamma[u] := \{(x, u(x)) : x \in \mathbb{R}\} \) of \( u \).
13.11. \textbf{Stieltjes measure (2). Stieltjes integrals.} 

(i) Show that $S[u] \in \mathcal{B}(\mathbb{R}^2)$.
(ii) Is it true that $\lambda^2(S[u]) = \int u \, d\lambda'$?
(iii) Show that $\Gamma[u] \in \mathcal{B}(\mathbb{R}^2)$ and that $\lambda^2(\Gamma[u]) = 0$.

[Hint: (i) – use T8.8 to approximate $u$ by simple functions $f_j \uparrow u$. Thus $S[u] = \bigcup_j S[f_j]$ and $S[f_j] \in \mathcal{B}(\mathbb{R}^2)$ is easy to see; alternatively, use T13.10, set $U(x, y) := (u(x), y)$ and observe that $S[u] = U^{-1}(C)$ for the closed set $C := \{(x, y) : x \geq y\}$; (ii) – use Tonelli’s theorem; (iii) – use $\Gamma[u] \subset S[u] \setminus S((u - \epsilon)^+)$ or $\Gamma[u] = U^{-1}((x, y) : x = y)$; show first that $\lambda^2(\Gamma[u] \cap [-n, n]^2) = 0$ for every $n \in \mathbb{N}$ and observe that $\Gamma[u] \cap [-n, n]^2 = \Gamma[(u_{[-n, n]}^\perp) \wedge n]$.

13.10. Let $(X, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space and let $u \in \mathcal{M}^+(\mathcal{A})$ be a $[0, \infty]$-valued measurable function. Show that the set

$$Y := \{y \in \mathbb{R} : \mu(\{x : u(x) = y\}) \neq 0\} \subset \mathbb{R}$$

is countable.

[Hint: assume that $u \in \mathcal{L}_1^\perp(\mu)$. Set $Y_{x, \eta} := \{y > \eta : \mu(\{u = y\}) > \epsilon\}$ and observe that for $t_1, \ldots, t_N \in Y_{x, \eta}$ we have $N \in \eta \leq \sum_{j=1}^N t_j \mu(\{u = t_j\}) \leq \int u \, d\mu$. Thus $Y_{x, \eta}$ is a finite set, and $Y = \bigcup_{k, n \in \mathbb{N}} Y_{k, \frac{1}{k}}$ is countable. If $u$ is not integrable, consider $(u \wedge m) 1_{A_m}$, $m \in \mathbb{N}$, where $A_m \uparrow X$ is an exhaustion.]

13.11. \textbf{Completion (5).} Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be any two measure spaces such that $\mathcal{A} \neq \mathcal{P}(X)$ and such that $\mathcal{B}$ contains non-empty null sets.

(i) Show that $\mu \times \nu$ on $(X \times Y, \mathcal{A} \otimes \mathcal{B})$ is not complete, even if both $\mu$ and $\nu$ were complete.

(ii) Conclude from (i) that neither $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \lambda \times \lambda)$ nor the product of the completed spaces $(\mathbb{R}^2, \mathcal{B}^*(\mathbb{R}) \otimes \mathcal{B}^*(\mathbb{R}), \lambda \times \lambda)$ are complete.

[Hint: you may assume in (ii) that $\mathcal{B}(\mathbb{R}), \mathcal{B}^*(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.]

13.12. Let $\mu$ be a bounded measure on the measure space $([0, \infty), \mathcal{B}[0, \infty))$.

(i) Show that $A \in \mathcal{B}[0, \infty) \otimes \mathcal{P}(\mathbb{N})$ if, and only if, $A = \bigcup_{j \in \mathbb{N}} B_j \times \{j\}$, where $(B_j)_{j \in \mathbb{N}} \subset \mathcal{B}[0, \infty)$.

(ii) Show that there exists a unique measure $\pi$ on $\mathcal{B}[0, \infty) \otimes \mathcal{P}(\mathbb{N})$ satisfying

$$\pi(B \times \{n\}) = \int_B e^{-t} \frac{t^n}{n!} \, d\mu(dt).$$

13.13. \textbf{Stieltjes measure (2). Stieltjes integrals.} This continues Problem 7.9. Let $\mu$ and $\nu$ be two measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu((-n, n]), \nu((-n, n]) < \infty$ for all $n \in \mathbb{N}$, and denote by

$$F(x) := \begin{cases} \mu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\mu((x, 0]), & \text{if } x < 0 \end{cases} \quad \text{and} \quad G(x) := \begin{cases} \nu((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -\nu((x, 0]), & \text{if } x < 0 \end{cases}$$

the associated right-continuous distribution functions (in Problem 7.9 we considered left-continuous distribution functions). Moreover, set $\Delta F(x) = F(x) - F(x-)$ and $\Delta G(x) = G(x) - G(x-)$. 

(i) Show that \( F, G \) are increasing, right-continuous and that \( \Delta F(x) = 0 \) if, and only if, \( \mu(\{ x \}) = 0 \). Moreover, \( F \) and \( \mu \) are in one-to-one correspondence.

(ii) Since measures and distribution functions are in one-to-one correspondence, it is customary to write \( \int u \, d\mu = \int u \, dF \), etc.

If \( a < b \) we set \( B := \{(x, y) : a < x \leq b, \, x \leq y \leq b\} \). Show that \( B \) is measurable and that

\[
\mu \times \nu(B) = \int_{(a,b]} F(s) \, dG(s) - F(a)(G(b) - G(a)).
\]

(iii) **Integration by parts.** Show that

\[
F(b)G(b) - F(a)G(a)
= \int_{[a,b]} F(s) \, dG(s) + \int_{(a,b]} G(s-) \, dF(s)
= \int_{[a,b]} F(s-) \, dG(s) + \int_{(a,b]} G(s-) \, dF(s) + \sum_{a < s \leq b} \Delta F(s) \Delta G(s).
\]

[Hint: expand \( \mu \times \nu((a, b])^2 \) in two different ways, using (ii). Note that the sum in the second part of the formula is at most countable because of L13.12.]

(iv) **Change of variable formula.** Let \( \phi \) be a \( C^1 \)-function. Then

\[
\phi(F(b)) - \phi(F(a))
= \int_{[a,b]} \phi'(F(s-)) \, dF(s) + \sum_{a < s \leq b} \left[ \phi(F(s)) - \phi(F(s-)) - \phi'(F(s)) \Delta F(s) \right].
\]

[Hint: use (iii) to show the change of variable formula for polynomials and then use the fact that continuous functions can be uniformly approximated by a sequence of polynomials – cf. Weierstraß’ approximation theorem 24.6.]

13.14. **Rearrangements.** Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and let \( f \in \mathcal{L}^p(\mu) \) for some \( p \in [1, \infty) \). The distribution function of \( f \) is given by \( \mu_f(\{ f \geq t \}) \) and the **decreasing rearrangement** of \( f \) is the generalized inverse of \( \mu_f \),

\[
f^*(\xi) := \inf \{ t : \mu_f(t) \leq \xi \}, \quad \xi \geq 0, \quad (\inf \emptyset = +\infty).
\]

(i) Let \( f = 2 \mathbf{1}_{[1,3]} + 4 \mathbf{1}_{[4,5]} + 3 \mathbf{1}_{[6,9]} \). Make a sketch of the graphs of \( f(x) \), \( \mu_f(t) \) and \( f^*(\xi) \).

(ii) Show that for \( f \in \mathcal{L}^p(\mu) \)

\[
\int_{\mathbb{R}} |f|^p \, d\mu = p \int_0^\infty t^{p-1} \mu_f(t) \, dt = \int_{(0,\infty)} (f^*)^p \, d\lambda.
\]

In other words: \( \| f \|_p = \| f^* \|_p \). Because of this the space \( \mathcal{L}^p \) is said to be **rearrangement invariant.**