Problem Set 3 - Do for Monday Oct. 16

Folland p. 123:
#23* (You may use without proof the fact that an open set in \( \mathbb{R} \) is a countable union of disjoint open intervals)
#29* (a*), (b) (Interesting) p. 127 #32*, #34*

Also:
1*. (Sequences are not enough) Let \( X = 2^\mathbb{R} \) be the product space, where 2 represents the two point set \{0, 1\}, with the discrete topology. Give \( X \) the product topology. Thus \( X \) can be identified with the set all functions from \( \mathbb{R} \) to \{0, 1\} (with no assumption of continuity). Let
\[
E = \{ f \in X : f(r) = 0 \text{ for all but finitely many } r \}.
\]
Show
(a) \( E \) is dense in \( X \)
(b) There is no sequence in \( E \) converging to the constant function \( f_1 \), where \( f_1(r) = 1 \ \forall r \in \mathbb{R} \).

2*. A topological space \((X, T)\) is totally disconnected if the connected component (see Exercise 10(d) of sec. 4.1) of every point \( x \in X \) is \( \{x\} \).
(a) Show that if \( T \) is the discrete topology, then \( X \) is totally disconnected.
(b) Show that if \( X \) is finite and totally disconnected, then \( T \) is the discrete topology.
(c) Give an example of \((X, T)\) totally disconnected, but where \( T \) is not the discrete topology.