Problem Set 5 - Do for Wednesday February 9

Can be handed in Friday, February 11

Folland, Chapter 5
p. 164: #30, #33*, #35, #39*

Also:

1*. Let \( \{f_n\} \) be a sequence of continuous functions from \( \mathbb{R} \) to itself such that \( \lim f_n(x) \) exists for all \( x \in \mathbb{R} \). Show that there exists an interval \( I \subset \mathbb{R} \) and \( C > 0 \) such that
\[
|f_n(x)| \leq C, \quad \forall x \in I, \quad \forall n \in \mathbb{N}
\]

2*. Suppose \( \{\Lambda_n\} \) is a sequence of bounded linear transformations from a normed linear space \( X \) to a Banach space \( Y \) satisfying:
(a) \( ||\Lambda_n|| \leq M < \infty \quad \forall n \)
(b) There is a dense subset \( E \subset X \) such that \( \{\Lambda_n(x)\} \) converges for all \( x \in E \). Prove that \( \{\Lambda_n(x)\} \) converges for all \( x \in X \).

3*. If \( X \) is a Banach space and \( \{x_n\} \) a sequence in \( X \) such that \( \{f(x_n)\} \) is bounded for every \( f \in X^* \), then \( \{x_n\} \) is bounded

Think about the following:

1. If \( X \) is a Banach space and \( \{x_n\} \) a sequence such that \( \{f(x_n)\} \) is Cauchy for every \( f \in X^* \), is \( \{x_n\} \) necessarily Cauchy? Answer is no, but you may need to look at Hilbert spaces for a counter-example.

2. If \( X \) is a Banach space and \( \{x_n\} \) a sequence such that there is an infinite dimensional Banach space \( Y \) with \( \{T(x_n)\} \) Cauchy for every \( T \in L(X,Y) \), is \( \{x_n\} \) necessarily Cauchy? (I don’t know the answer to this one!)