1. (a) (10 points) Find the general solution to the differential equation
\[ t^3 y' + 4t^2 y = e^{-t}. \]
Rewrite as \( y' + \frac{4}{t} y = \frac{e^{-t}}{t^3} \). This is a linear first-order equation, so next compute the integrating factor \( \mu = e^{\int \frac{4}{t} \, dt} = e^{4 \ln t} = t^4 \). Multiply through by \( t^4 \) to get \( t^4 y' + 4t^3 y = te^{-t} \). The LHS is \( \frac{d}{dt}(t^4 y) \), so taking the antiderivative of both sides gives \( t^4 y = -te^{-t} - e^{-t} + C \). So the general solution is
\[ y(t) = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}. \]

(b) (8 points) Find the particular solution to the differential equation in part (a) with initial value \( y(1) = 0 \).
\[ y(1) = -e^{-1} - e^{-1} + C = 0, \text{ so } C = 2e^{-1}. \] The solution is
\[ y(t) = -\frac{e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{2e^{-1}}{t^4}. \]

(c) (2 points) What is the largest interval on which the solution you found in part (b) is defined?
The solution is defined on the interval \((0, \infty)\).

2. Consider the autonomous differential equation
\[ \frac{dy}{dt} = f(y), \]
where \( f(y) = (9 - y^2)(4 - y^2) \).

(a) (10 points) Draw a phase line, and label each equilibrium solution as stable, unstable, or semistable.
The equilibrium solutions are \( y = -3 \), \( y = -2 \), \( y = 2 \), and \( y = 3 \). \( y = 2 \) and \( y = -3 \) are stable, and \( y = 3 \) and \( y = -2 \) are unstable.

(b) (10 points) What is the long-term behavior of the solution satisfying \( y(0) = 4 \)?
If \( y(0) = 4 \), then \( y(t) \to \infty \) as \( t \to \infty \).

3. A tank is filled with 1000 L of brine, containing 15 kg of dissolved salt. Fresh water flows into the tank at a rate of 5 L/min. The solution is kept completely mixed, so the concentration of salt is uniform, and the solution flows out at 5 L/min.
(a) (8 points) Set up a differential equation that gives the rate of change of the amount of salt in the tank at any time $t$.

\[
\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out}) = 0 - 5 \cdot \frac{Q}{1000} = -0.005Q, \quad Q(0) = 15
\]

(b) (8 points) Solve the differential equation you found in part (a).

\[Q = 15e^{-0.005t}\]

(c) (4 points) What is the limiting amount of salt in the tank as $t \to \infty$?

\[Q \to 0 \quad \text{as} \quad t \to \infty\]

(d) (5 points) Suppose that instead of fresh water, a salt mixture with concentration 0.01 kg/L flows in at a rate of 3 L/min. Suppose also that the outflow remains at 5 L/min, so that the volume in the tank is no longer constant. Set up (but do not solve) the differential equation for the rate of change of the amount of salt in the tank at time $t$.

Note that the volume is decreasing at a rate of 2 L/min.

\[
\frac{dQ}{dt} = 0.03 - 5 \cdot \frac{Q}{1000-t}
\]

4. (a) (7 points) Find a fundamental set of solutions to the differential equation

\[y'' + 5y' + 6y = 0.\]

\[r^2 + 5r + 6 = 0\]

\[(r + 3)(r + 2) = 0\]

Fundamental set: \(y_1(t) = e^{-3t}, \quad y_2(t) = e^{-2t}\)

(b) (3 points) Check that the solutions you found in part (a) are linearly independent.

\[
\text{Compute the Wronskian:} \quad W(y_1, y_2) = \begin{vmatrix} e^{-3t} & e^{-2t} \\ -3e^{-3t} & -2e^{-2t} \end{vmatrix} = e^{-5t}.
\]

Since the Wronskian is non-zero, \(y_1\) and \(y_2\) are linearly independent.

(c) (10 points) Solve the initial value problem

\[y'' + 5y' + 6y = 0, \quad y(0) = 2, \quad y'(0) = -1.\]

\[y(t) = c_1e^{-3t} + c_2e^{-2t}\]

\[y'(t) = -3c_1e^{-3t} - 2c_2e^{-2t}\]

\[y(0) = c_1 + c_2 = 2\]

\[y'(0) = -3c_1 - 2c_2 = -1\]

\[c_1 = -3, \quad c_2 = 5\]

\[y(t) = -3e^{-3t} + 5e^{-2t}\]