Practice Problems for Final
Math 20D, F07

1. (i) State the order of the given differential equation. (ii) Classify the
differential equation as ordinary or partial and (iii) linear or nonlinear.

(a) \( \frac{d^2y}{dt^2} + \sin(t + y) = \cos t \)

Solution: order 2, ODE, nonlinear

(b) \( \frac{\partial^3 y}{\partial t^3} + \frac{\partial y}{\partial t} = 2e^t y + t \)

Solution: order 3, PDE, linear

2. Determine whether each of the given functions is a solution of the differ-
eential equation:

\( t^2 y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2} \ln(t), \quad y_2(t) = 2t^{-4} \)

Solution: \( y_1 \) is a solution, but \( y_2 \) is not.

3. Find the solution of the given initial value problem in explicit form:

\( y' - y = 2te^{2t}, \quad y(0) = 1 \)

Solution: Using the method of integrating factors, one obtains the solution

\( y(t) = 2te^{2t} - 2e^{2t} + 3e^t \).

4. Find the solution of the given initial value problem in explicit form.
Also determine the interval on which the solution is valid:

\( y' = (1 - 2x)y^2, \quad y(0) = -\frac{1}{6} \)
Solution: The differential equation is separable. The solution is

\[ y = \frac{1}{x^2 - x - 6} \]

with interval of validity \((-2, 3)\).

5. Sketch several solutions to the given autonomous differential equation in the \(t-y\) plane. Determine the equilibrium solutions and classify each as asymptotically stable, unstable, or semistable.

\[ \frac{dy}{dt} = (y^2 - 16)(y^2 - 4y + 4) \]

Are there any initial conditions which guarantee that a solution will remain finite in the long-term?

Solution: (Sketch not included) \(y = 2\) is a semistable equilibrium solution, \(y = 4\) is an unstable equilibrium solution, \(y = -4\) is a stable equilibrium solution. A solution \(y\) will remain finite in the long-term if and only if \(\infty < y(0) \leq 4\).

6. Solve the given differential equation:

\[ y' = -\frac{2x + 3y}{3x + 4y} \]

Solution: The differential equation is exact. The general solution (in implicit form) is

\[ 2y^2 + x^2 + 3xy = C. \]

7. Find the general solution of the given differential equation:

(a) \(9y'' - 12y' + 4y = 0\)

Solution: \(y(t) = c_1 e^{\frac{2t}{3}} + c_2 te^{\frac{2t}{3}}.\)
(b) \(9y'' + 6y' + 82y = 0\)

Solution: \(y(t) = c_1 e^{\frac{-t}{3}} \cos(3t) + c_2 e^{\frac{-t}{3}} \sin(3t)\).

(c) \(y'' - 3y = 0\)

Solution: \(y(t) = c_1 e^{\sqrt{3} t} + c_2 e^{-\sqrt{3} t}\).

8. Determine a suitable form for the particular solution \(Y_p\) of the following given differential equation or system if you were to solve it using the method of undetermined coefficients. (Don’t solve though.)

(a)
\[y'' - 3y' - 4y = 2e^{-t} + t \cos(2t)\]

Solution: \(Y_p = Ate^{-t} + (Bt + C)(E \cos(2t) + F \sin(2t))\)

(b)
\[
\vec{x}' = \begin{pmatrix}
2 & -1 \\
3 & -2
\end{pmatrix} \vec{x} + \begin{pmatrix}
e^t \\
t
\end{pmatrix}
\]

Solution: \(\vec{Y}_p = \tilde{a}e^t + \tilde{b}e^t + \tilde{c}t + \tilde{d}\). Note: Sorry guys, I forgot that we didn’t cover this type of problem this year because of the fire. Don’t worry about this one.

9. Consider the linear system

\[
\frac{dx}{dt} = x + 2y \nonumber
\]
\[
\frac{dy}{dt} = -5x - y
\]

Write the system in matrix form, solve the system, and sketch the phase portrait.

Solution: The system can be written as

\[
\vec{x}' = \begin{pmatrix}
1 & 2 \\
-5 & -1
\end{pmatrix} \vec{x}.
\]
The general solution is
\[ \vec{y}(t) = c_1 \left( \begin{array}{c} 2 \cos(3t) \\ -\cos(3t) - 3 \sin(3t) \end{array} \right) + c_2 \left( \begin{array}{c} 2 \sin(3t) \\ 3 \cos(3t) - \sin(3t) \end{array} \right). \]

The phase portrait consists of ellipses centered at the origin with trajectories moving in the clockwise direction.

10. Recall that Newton’s Law of Cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the ambient temperature.

(a) Suppose that the ambient temperature is 15°C and the rate constant is 0.1 (hr)$^{-1}$. Write the differential equation implied by Newton’s Law of Cooling.

Solution: \[ \frac{dT}{dt} = 0.1(15 - T) \]

(b) Solve the differential equation given that the initial temperature of the object is 21°C.

Solution: \[ T = 15 + 6e^{-0.1t} \]

11. Find the recurrence relation satisfied by the coefficients $a_n$ of the power series solution of the following differential equation about the ordinary point $x_0 = 0$. Also find the first four terms in each of two linearly independent solutions.

\[ y'' - xy' - y = 0. \]

Solution: The recurrence relation is
\[ a_{n+2} = \frac{a_n}{n+2}, \quad n = 0, 1, 2, \ldots. \]

Two linearly independent solutions are:
\[ y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots \]
\[ y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots \]
12. What can you say about the radius of convergence of series solutions about each given point $x_0$ for the given differential equation?

$$(x^2 - 2x + 5)y'' + xy' + 5y = 0, \quad x_0 = 1, \quad x_0 = 0$$

Solution: The radius of convergence for a series solution about $x_0 = 1$ is at least 2, and the radius of convergence for a series solution about $x_0 = 0$ is at least $\sqrt{5}$.

13. Use the Laplace transform to solve the given IVP.

(a) $$y'' - y' = \cos(2t) + u_6(t) \cos(2t - 12), \quad y(0) = -4, \quad y'(0) = 0$$

Solution: Taking the Laplace transform of the differential equation and using partial fraction decomposition, we find that

$$Y(s) = \frac{1 + e^{-6s}}{5} \left( \frac{1}{s-1} - \frac{s+1}{s^2+4} \right) - \frac{4}{s}. $$

Thus the solution of the IVP is

$$y(t) = \frac{1}{5} \left( e^t - \cos(2t) - \frac{1}{2} \sin(2t) \right) + \frac{1}{5} u_6(t) \left( e^{t-6} - \cos(2t - 12) - \frac{1}{2} \sin(2t - 12) \right) - 4$$

(b) $$y'' + 3y' + 2y = \delta(t-5) + u_{10}(t), \quad y(0) = 0, \quad y'(0) = 1/2$$

Solution: Taking the Laplace transform of the differential equation and using partial fraction decomposition, we find that

$$Y(s) = \left( e^{-5s} + \frac{1}{2} \right) \left( \frac{1}{s+1} - \frac{1}{s+2} \right) + \frac{1}{2} e^{-10s} \left( \frac{-2}{s+1} + \frac{1}{s+2} + \frac{1}{s} \right).$$

Thus the solution of the IVP is

$$y(t) = \frac{1}{2} \left( e^{-t} - e^{-2t} \right) + u_5(t) \left( e^{-(t-5)} - e^{-2(t-5)t} \right) + \frac{1}{2} u_{10}(t) \left( -2e^{-(t-10)} + e^{-2(t-10)} + 1 \right)$$
14. If the Wronskian of $f$ and $g$ is $3e^{4t}$ and $f(t) = e^{2t}$, find $g(t)$.

Solution: We have $e^{2t}(g' - 2g) = 3e^{4t}$. Using the method of integrating factors, we find that

$$g(t) = 3te^{2t} + Ce^{2t}.$$