100A Fall 2010 - Subgroups of $S_4$

The dihedral group on 4 elements

(1) What is the physical interpretation of the elements of $D_4$ in terms of rigid motions?

The rigid motions of a square.

(2) In class, we showed that for any $n \geq 3$, the dihedral group $D_n$ is the collection of all products of powers of $a$ and $b$ where $a$ corresponds to counterclockwise rotation by $\frac{360}{n}^\circ$ and $b$ corresponds to a flip across the vertical axis.

We use the convention that the positions of vertices of the square are labelled starting at the bottom right corner and increasing counterclockwise.

(a) Write $a \in D_4$ in cycle notation.

$$a = (1, 2, 3, 4)$$

(b) Write $b \in D_4$ in cycle notation.

$$b = (1, 4)(2, 3)$$

(c) Write the permutation corresponding to a clockwise rotation by $90^\circ$ first in cycle notation and then as a product of powers of $a$ and $b$.

$$\sigma = (1, 4, 3, 2).$$

To write this as a product of powers of $a$ and $b$, notice that a clockwise rotation by $90^\circ$ is the same as a counterclockwise rotation by $270^\circ$. So,

$$\sigma = a^3.$$

To confirm this answer, we can compute:

$$a^2 = (1, 2, 3, 4)(1, 2, 3, 4) = (1, 3)(2, 4)$$

$$a^3 = (1, 2, 34)[(1, 3)(2, 4)] = (1, 4, 3, 2) = \sigma.$$

(3) What is the order of $D_4$?

Using the identities $o(a) = 4$ and $o(b) = 2$ and $ab \neq ba$, we can show that

$$|D_4| = 4 \cdot 2 = 8.$$

(4) Is $D_4 \cong Z_4 \times Z_2$?

No. $D_4$ is not abelian: $ab \neq ba$ but $Z_4 \times Z_2$ is abelian since it is the direct product of two abelian groups.

$Figure 3.6.6 in Beachy and Blair shows the subgroup diagram of D_4.$

The alternating group on 4 elements

(1) What is the order of $A_4$?

We showed in class that the order of $A_n$ is $\frac{n!}{2}$. So,

$$|A_4| = \frac{4!}{2} = \frac{24}{2} = 12.$$
(2) List out all the elements of $A_4$ in cycle notation.

Each element of $A_4$ is an even permutation of 4 elements. Therefore, it is either a cycle of length 1, a cycle of length 3, or a product of two disjoint transpositions.

\[
\begin{align*}
(1), \\
(1 \ 2 \ 3), (1 \ 2 \ 4), (1 \ 3 \ 2), (1 \ 3 \ 4), (1 \ 4 \ 2), (1 \ 4 \ 3), (2 \ 3 \ 4), (2 \ 4 \ 3), \\
(1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3)
\end{align*}
\]

(3) Show that $S_2$ is isomorphic to a subgroup of $A_4$.

Since $S_2$ has order 2 and we showed that any two groups of order 2 are isomorphic, it’s sufficient to exhibit a subgroup of $A_4$ of order 2. For example, consider the subset \{(1), (1 \ 2)(3 \ 4)\}. Then

\[
[(1 \ 2)(3 \ 4)][(1 \ 2)(3 \ 4)] = (1)
\]

so $o((1 \ 2)(3 \ 4)) = 2$ and this subset is the subgroup of $A_4$ generated by $(1 \ 2)(3 \ 4)$, hence a subgroup.