Prove Proposition 3.7.6: Let $\varphi : G_1 \to G_2$ is a group homomorphism.

If $H_1$ is a normal subgroup of $G_1$ and $\varphi$ is onto, prove that $\varphi(H_1)$ is normal in $G_2$.

Suppose $\varphi$ is onto and $H_1$ is normal in $G_1$. To show that $\varphi(H_1)$ is normal in $G_2$, assume $y \in \varphi(H_1)$ and $b \in G_2$. Then (since $\varphi$ is onto), there is $h \in H_1$ and $a \in G_1$ such that

$$y = \varphi(x), \quad b = \varphi(a).$$

So

$$byb^{-1} = \varphi(axa^{-1}) \in \varphi(H_1) \quad \text{because} \quad axa^{-1} \in H_1.$$  

If $H_2$ is a normal subgroup of $G_2$ then prove that $\varphi^{-1}(H_2)$ is a normal subgroup of $G_1$.

First we show that $\varphi^{-1}(H_2)$ is a subgroup of $G_1$. $\varphi^{-1}(H_2)$ is nonempty: $e_1 \in \varphi^{-1}(H_2)$ because $e_2 \in H_2$ and $\varphi$ is a homomorphism. Suppose $a, b \in \varphi^{-1}(H_2)$. There are $x, y \in H_2$ such that $\varphi(a) = x, \varphi(b) = y$. Using properties of homomorphisms and the fact that $H_2$ is a subgroup,

$$\varphi(ab^{-1}) = xy^{-1} \in H_2 \quad \text{so} \quad ab^{-1} \in \varphi^{-1}(H_2).$$

Assume now that $H_2$ is normal in $G_2$ and we show that $\varphi^{-1}(H_2)$ is normal in $G_1$. Let $a \in G_1$ and $h \in \varphi^{-1}(H_2)$. Let $h' \in H_2$ be such that $\varphi(h) = h'$. Then

$$\varphi(aha^{-1}) = \varphi(a)h'\varphi(a)^{-1} \in H_2 \quad \text{because} \quad H_2 \text{ is normal.}$$

Thus, $aha^{-1} \in \varphi^{-1}(H_2)$ as required.
List all the left cosets of the subgroup $4\mathbb{Z}$ in the (additive) group $\mathbb{Z}$.

Each left coset is determined by some integer $a$: $a + 4\mathbb{Z}$ (notice the additive notation).

- $a = 0$: $0 + 4\mathbb{Z} = 4\mathbb{Z}$.
- $a = 1$: $1 + 4\mathbb{Z} = \{1, 5, 9, \ldots, -3, -7, \ldots\}$.
- $a = 2$: $2 + 4\mathbb{Z} = \{2, 6, 10, \ldots, -2, -6, \ldots\}$.
- $a = 3$: $3 + 4\mathbb{Z} = \{3, 7, 11, \ldots, -1, -5, -9, \ldots\}$.

There are no more left cosets. Why? Suppose $a \in \mathbb{Z}$ is not $0, 1, 2$ or $3$. Then by the division algorithm there are $q \in \mathbb{Z}$ and $0 \leq r < 4$ such that $a = 4q + r$. This implies that $a \in r + 4\mathbb{Z}$. We have proved that if $a \in r + 4\mathbb{Z}$ then $a + 4\mathbb{Z} = r + 4\mathbb{Z}$. Thus, the left coset of $4\mathbb{Z}$ in $\mathbb{Z}$ determined by $a$ is already represented in the list above.

Without doing any more computation, list all the right cosets of $4\mathbb{Z}$ in $\mathbb{Z}$.

*Hint: use the fact that $\mathbb{Z}$ is abelian.*

Since $\mathbb{Z}$ is abelian, each left coset is a right coset. Why? For any $x, a \in \mathbb{Z}$, $x \in a + 4\mathbb{Z}$ if and only if there is $h \in 4\mathbb{Z}$ such that $x = a + h$. But, this happens exactly when $x = h + a$, which is the condition for $x \in 4\mathbb{Z} + a$. Thus, the list of right cosets of $4\mathbb{Z}$ in $\mathbb{Z}$ is the same as that of left cosets.

What is $[\mathbb{Z} : 4\mathbb{Z}]$?

By definition, this is the number of (distinct) left cosets of $4\mathbb{Z}$ in $\mathbb{Z}$. By the above list, we see that $[\mathbb{Z} : 4\mathbb{Z}] = 4$.

What does this tell us about the isomorphism type of $\mathbb{Z}/4\mathbb{Z}$?

First, notice that $|\mathbb{Z}/4\mathbb{Z}| = [\mathbb{Z} : 4\mathbb{Z}]$. Why? The set underlying the group $\mathbb{Z}/4\mathbb{Z}$ is the set of left cosets of $4\mathbb{Z}$ in $\mathbb{Z}$. And, $[\mathbb{Z} : 4\mathbb{Z}]$ is defined to equal the number of such cosets. Thus, the order of the group $\mathbb{Z}/4\mathbb{Z}$ is equal to the index of $4\mathbb{Z}$ in $\mathbb{Z}$.

We have shown that any group of order 4 is isomorphic either to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Moreover, we can prove that $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$ by exhibiting an element of $\mathbb{Z}/4\mathbb{Z}$ of order 4. Recall that the identity in $\mathbb{Z}/4\mathbb{Z}$ is the coset determined by $0$, $\mathbb{Z}$. Consider the coset $1 + 4\mathbb{Z}$.

Then

$$
1(1+4\mathbb{Z}) = (1+4\mathbb{Z}) \neq \mathbb{Z} \quad 2(1+4\mathbb{Z}) = (2+4\mathbb{Z}) \neq \mathbb{Z} \quad 3(1+4\mathbb{Z}) = (3+4\mathbb{Z}) \neq \mathbb{Z} \quad 4(1+4\mathbb{Z}) = \mathbb{Z}.
$$

Thus, $o(1 + 4\mathbb{Z}) = 4$ and so $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$.

*In general, one can prove that $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ for any positive $n$.*