1. Through this question, you will classify all groups of order less than or equal to 7 up to isomorphism. This means that for each $1 \leq n \leq 7$, you will find a list of groups of order $n$ such that any other group of order $n$ must be isomorphic to some group in this list.

(a) Show that there is a unique group of order 1 up to isomorphism.

(b) Show that there is a unique group of any prime order up to isomorphism. Thus, deduce that for $n = 2, 3, 5, 7$, there is a unique group of order $n$ up to isomorphism.

(c) To classify groups of order 4, show that any group of order 4 is isomorphic either to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) To classify groups of order 6, use your work from questions 15 (week 4), 16, and the hint from question 17 in section 3.3 of Beachy and Blair.

2. Fix some $n \geq 2$. In this problem, you will find all subgroups of the group $\mathbb{Z}_n$. As a consequence, you will prove that every subgroup of $\mathbb{Z}_n$ is cyclic, equal to $\langle [d]_n \rangle$ for some positive divisor $d$ of $n$.

(a) Show that if $d \geq 1$ and $d|n$, then $\langle [d]_n \rangle$ has exactly $\frac{n}{d}$ elements. Describe these elements.

(b) Let $H$ be any subgroup whatsoever of $\mathbb{Z}_n$ (don’t assume $H$ is cyclic). Show that there must be a positive integer $m$ such that $[m]_n \in H$. Thus it makes sense to let $d$ be the smallest positive integer such that $[d]_n \in H$, using the well-ordering principle. Prove that $d|n$.

(c) Again let $H$ be any subgroup of $\mathbb{Z}_n$ and let $d$ be defined as in part (b). Prove that $H$ is equal to the cyclic subgroup $\langle [d]_n \rangle$ of $\mathbb{Z}_n$.

*Hint: suppose that $[c]_n \in H$. Use the division algorithm to write $c = qd + r$ for some $q, r$. Show that $[r]_n \in H$, and so...*