100A: Abstract Algebra 1
Addendum to Homework for Week 9
Posted Nov 22, 2010

The solutions to these questions are to be submitted in addition to the questions from the textbook (Beachy and Blair) posted online. The homework is due by the beginning of lecture on November 29, 2010.

For each $n \in \mathbb{N}^>0$, we define the **orthogonal group**\(^1\) to be

$$O_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^T = A^{-1}\} = \{A \in GL_n(\mathbb{R}) : AA^T = A^TA = Id\}$$

where the **transpose** of a matrix is defined as the result of interchanging columns and rows. That is, if

$$A = (a_{ij}) \quad \text{then} \quad A^T = (a_{ji}).$$

For example,

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

One can prove that any matrix in $O_2(\mathbb{R})$ has determinant equal to $\pm 1$ and that $O_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$. (You may use these facts in your solutions.)

We define the **special orthogonal group** of dimension $n$, $SO_n(\mathbb{R})$, to be the subset of $O_n(\mathbb{R})$ with determinant equal to 1:

$$SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) : \det(A) = 1\} = \{A \in SL_n(\mathbb{R}) : A^T = A^{-1}\} = \{A \in SL_n(\mathbb{R}) : AA^T = A^TA = Id\}$$

1. Prove that $SO_2(\mathbb{R})$ is a subgroup of $O_2(\mathbb{R})$.

2. What is the left coset of $SO_2(\mathbb{R})$ in $O_2(\mathbb{R})$ determined by

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}?$$

3. Is $SO_2(\mathbb{R})$ a normal subgroup of $O_2(\mathbb{R})$? *Hint: you may wish to compute $[O_2(\mathbb{R}) : SO_2(\mathbb{R})]$.*

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\(^1\)When $n = 2$, this group corresponds exactly to the set of reflections and rotations of the plane.