100A Fall 2010 - Quiz 3

(1) What is the order of \((\mathbb{Z}_{96}, +)\)?

(2) What is the order of \((\mathbb{Z}_{96}^\times, \cdot)\)?
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\[\mathbb{Z}_{96} = \{[0]_{96}, [1]_{96}, \ldots, [95]_{96}\}.$ Hence, the group has order 96.

(2) What is the order of \((\mathbb{Z}_{96}^\times, \cdot)\)?

\((\mathbb{Z}_{96}^\times, \cdot)\) is the group of units of \(\mathbb{Z}_{96}\). A positive number \(a \leq 96\) represents a unit if and only if \((a, 96) = 1\). Euler’s phi function counts the number of such \(a\). To compute \(\varphi(96)\), it is useful to find the prime factorization of 96.

\[96 = 3^2 \cdot 3^3 = 2^5 \cdot 3^2.\]

Hence,

\[
\varphi(96) = \varphi(2^5 \cdot 3) = 96 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 96 \left(\frac{1}{2}\right) \left(\frac{2}{3}\right) = \frac{2^5 \cdot 3}{3} = 32.
\]

Therefore, the order of \((\mathbb{Z}_{96}^\times, \cdot)\) is 32.