Final Review

1) What is a group?

2) Let * be defined on \( \mathbb{Z} \) by \( a*b = a-b \). Is \((\mathbb{Z}, *)\) a group?

3) List all isomorphism types of groups up to order 7.

4) Let \( H \subseteq G \) be nonempty with the property that \( a, b \in H \) implies \( a*b \in H \). Under what additional assumption is \( H \) a subgroup of \( G \)?

5) Let \( \phi: \mathbb{Z}_n \to \mathbb{Z}_n \) be a homomorphism. Suppose \( \phi([1]) = [a]_n \). What are the possible values of \( a \)? Why does \([a]_n\) determine \( \phi \)?

6) Let \( D = \{ f = mx + b \text{ from } \mathbb{R} \to \mathbb{R} | \text{int} x = 0, 3, 6 \} \), and \( D \) the subset of elements of \( B \) with \( m \neq 0 \) that is, \( D = \{ f = mx + b | \text{int} x = 3 \} \). Is \( D \) a subgroup? Is \( D \) normal? Describe \( B/D \).

7) Define \( \phi: GL_n(\mathbb{R}) \to \mathbb{R}^* \) by \( \phi(A) = \det A \). Why is \( \phi \) a homomorphism? What is the kernel of \( \phi \)? Assuming \( \phi \) is onto, what extra isomorphism is determined by \( \phi \)?
9) Let \( H \) be a subgroup of \( G \). How can you tell if \( a, b \in G \) are in the same left coset of \( H \)?

10) Define \([G:N]\). How can it be computed if \( G \) is finite?

11) Fill in implication arrows:
   - \( 1 \) : prime, \( G \) cyclic, \( G \) abelian, \( \text{Center of } G = G \)
   - All subgroups are normal

12) Describe, for \( g, h \in G \), \( ghg^{-1} \) in terms of \( g, h, g^{-1} \).

13) List three conditions showing \( N \trianglelefteq G \) (\( N \) normal in \( G \)).

14) Let \( N \trianglelefteq G \). Give a homomorphism with \( \ker \varphi = N \).

15) Let \( \varphi : G \to H \) be a homomorphism. Construct an isomorphism.

16) Describe \( \text{Z}(G) \) (center) in terms of centralizers.

17) Describe the first isomorphism theorem in a picture.
13) Construct a bijection from \( D_n \) to \( \mathbb{Z}_n \times \mathbb{Z}_2 \). Is this a homomorphism?

Tips for exam:
- Be careful to state starting information - assumptions and definitions. Then write the conclusion you are hoping to arrive at. Fill in the middle stuff.
- Use the FTH (\( G/\ker \phi \cong \im \phi \))
- To show \( S \) is equal to another set, \( T \), show \( S \subseteq T \) and \( T \subseteq S \). This can be useful in finding kernels, show some set \( H = \ker \phi \).
- Take your time
- Draw a picture if the situation is unclear.
- Try a simple example to get intuition.
- Remember where your elements live. For example, if \( ab \in \mathbb{H} \), look at \( ab \in \mathbb{E} \).
- For small groups, multiplication tables can be informative.