Homework 8

Section 3.6: 2 (only for $\mathbb{Z}_2 \times \mathbb{Z}_2$), 17, 21, 26;
Section 3.7: 7 (b), (d), and (f), 9, 14, 15.

Section 3.6:

2. Write out the addition tables for $\mathbb{Z}_4$ and for $\mathbb{Z}_2 \times \mathbb{Z}_2$. Use cycle notation to write out the permutation determined by each row of each of the addition tables, as in the discussion preceding Cayley’s theorem.

\[
\begin{array}{c|cccc}
Z_4: & + & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\]

The associated permutations, row by row, are $(1)$, $(0, 1, 2, 3)$, $(0, 2)(1, 3)$, $(0, 3, 2, 1)$.

\[
\begin{array}{c|cccc}
\mathbb{Z}_2 \times \mathbb{Z}_2: & + & (0, 0) & (1, 0) & (0, 1) & (1, 1) \\
\hline
(0, 0) & (0, 0) & (1, 0) & (0, 1) & (1, 1) \\
(1, 0) & (1, 0) & (0, 0) & (1, 1) & (0, 1) \\
(0, 1) & (0, 1) & (1, 1) & (0, 0) & (1, 0) \\
(1, 1) & (1, 1) & (0, 1) & (1, 0) & (0, 0) \\
\end{array}
\]

Letting $(0, 0) = 1$, $(1, 0) = 2$, $(0, 1) = 3$, and $(1, 1) = 4$, the permutations associated with the rows of the above table are $(1)$, $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, $(1, 4)(2, 3)$.

17. For any elements $\sigma, \tau \in S_n$, show that $\sigma \tau \sigma^{-1} \tau^{-1} \in A_n$.

Write $\sigma$ and $\tau$ each as a product of transpositions, containing $s$ and $t$ terms, respectively. Then $\sigma^{-1}$ and $\tau^{-1}$ can be written as a product of $s$ and $t$ transpositions. Hence $\sigma \tau \sigma^{-1} \tau^{-1}$ can be written as a product of $s + t + s + t = 2s + 2t$ transpositions. Since $2(s + t)$ is even, $\sigma \tau \sigma^{-1} \tau^{-1} \in A_n$.

21. Find the center of the dihedral group $D_n$.

Hint: Consider two cases, depending on whether $n$ is odd or even.

Let $n \geq 3$. Then $D_n = \{a^j, a^j b \mid 0 \leq j < n\}$ with $a^n = b^2 = e$ and $ba = a^{n-1}b$. By induction we have $ba^j = a^{n-j}b$. Now if $x = a^j b$ then $xa = a^j ba = a^{j+n-1}b$ and $ax = a^{j+1}b$. Hence $xa = ax$ if and only if $j + n - 1 = j + 1 \pmod{n}$, and this happens if and only if $n = 2$. Hence $x = a^j b$ is never central.

Now let $x = a^j$. Since $x$ commutes with all powers of $a$, it will be central if $xb = bx$. But $bx = ba^j = a^{n-j}b = a^j b$ if and only if $n - j \equiv j \pmod{n}$. This holds for $0 \leq j < n$ only if $j = 0$ or $j = m$ when $n = 2m$. Thus if $n = 2m$, then $Z(D_n) = \{e, a^m\}$ and if $n$ is odd, then $Z(D_n) = \{e\}$. 
26. Prove that every group of order \( n \) is isomorphic to a subgroup of \( \text{GL}_n(\mathbb{R}) \).

We will show that \( S_n \) is a subgroup of \( \text{GL}_n(\mathbb{R}) \), and then apply Cayley’s theorem.

For \( \sigma \in S_n \), let \( a_\sigma = (a_{ij}) \) be the \( n \times n \) matrix for which \( a_{ij} = \delta_{\sigma^{-1}i,j} \), where \( \delta \) is the Kronecker delta. Note that \( a_\sigma \in \text{GL}_n(\mathbb{R}) \) since 1 appears exactly once in each row and column of \( a_\sigma \). Define \( \phi : S_n \to \text{GL}_n(\mathbb{R}) \) by \( \phi(\sigma) = a_\sigma \). To see that \( \phi \) is a homomorphism, let \( \phi(\sigma) = (a_{ij}) \) and \( \phi(\tau) = (b_{ij}) \).

The product matrix \((c_{ij})\) is determined by \( c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \). The element \( c_{ij} \) is \( \mathbb{I} \) if and only if \( k = \sigma^{-1}(i) \) and \( j = \tau^{-1}(k) \), if and only if \( j = \tau^{-1}(\sigma^{-1}(i)) = (\sigma\tau)^{-1}(i) \). Thus \( \phi(\sigma)\phi(\tau) = \phi(\sigma\tau) \). Next note that \( \phi(\sigma) \) is the identity matrix if and only if \( i = \sigma^{-1}(i) \) for all \( i \), if and only if \( \sigma \) is the identity permutation. Hence \( \phi \) is one-to-one.

Since by Cayley’s theorem \( G \) is isomorphic to a subgroup of \( S_n \) and \( S_n \) is isomorphic to a subgroup of \( \text{GL}_n(\mathbb{R}) \) by the last paragraph, \( G \) is isomorphic to a subgroup of \( \text{GL}_n(\mathbb{R}) \).

Section 3.7

7. Which of the following functions are homomorphisms?

\( \uparrow (a) \) \( \phi : \mathbb{R}^\times \to \text{GL}_2(\mathbb{R}) \) defined by \( \phi(a) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \)

We have \( \phi(ab) = \begin{bmatrix} ab & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} = \phi(a)\phi(b) \), so \( \phi \) is a homomorphism.

\( \uparrow (b) \) \( \phi : \mathbb{R} \to \text{GL}_2(\mathbb{R}) \) defined by \( \phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \)

We have \( \phi(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \phi(a)\phi(b) \), so \( \phi \) is a homomorphism.

\( \uparrow (c) \) \( \phi : \text{M}_2(\mathbb{R}) \to \mathbb{R} \) defined by \( \phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a \)

Since

\[ \phi \left( \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) = \phi \left( \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \right) = a_1 + a_2 \]

we conclude that \( \phi \) is a homomorphism.
(d) $\phi : \text{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ab$

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$, and $\phi(A^2) = 6$ but $\phi(A)\phi(A) = 3 \cdot 3 = 9$, and so $\phi$ is not a homomorphism.

(e) $\phi : \text{GL}_2(\mathbb{R}) \to \mathbb{R}$ defined by $\phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a + d$

Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\phi(I \cdot I) = \phi(I) = 2$, but $\phi(I) + \phi(I) = 2 + 2 = 4$, and so $\phi$ is not a homomorphism.

(f) $\phi : \text{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$

The mapping $\phi$ is a homomorphism since $\phi(AB) = \det A \det B$ for all $A, B \in \text{GL}_2(\mathbb{R})$.

9. Let $\phi$ be a group homomorphism of $G_1$ onto $G_2$. Prove that if $G_1$ is abelian then so is $G_2$; prove that if $G_1$ is cyclic then so is $G_2$. In each case, give a counterexample to the converse of the statement.

Suppose that $G_1$ is abelian. If $a, b \in G_2$, then there exist $c, d \in G_1$ such that $a = \phi(c)$ and $b = \phi(d)$. Then $ab = \phi(a)\phi(b) = \phi(c\phi(d)) = \phi(cd) = \phi(d\phi(c)) = ba$. Hence $G_2$ is abelian.

Suppose that $G_1 = \langle a \rangle$ is cyclic, and let $y \in G_2$. Then $y = \phi(x)$ for some $x \in G_1$, and since $G_1 = \langle a \rangle$ we have $x = a^n$ for some $n \in \mathbb{Z}$. Thus $y = \phi(x) = \phi(a^n) = (\phi(a))^n \in \langle \phi(a) \rangle$. Hence $G_2 \subseteq \langle \phi(a) \rangle$.

Clearly $\langle \phi(a) \rangle \subseteq G_2$ and so $G_2 = \langle \phi(a) \rangle$ is cyclic.

Let $G_1 = \text{GL}_2(\mathbb{Z}_3)$. Then $G_1$ is not abelian and so not cyclic. Define $\phi : G_1 \to \mathbb{Z}_4$ by $\phi(A) = \det A$. Since $\mathbb{Z}_3$ is cyclic and abelian we have our desired counterexample.

14. Recall that the center of a group $G$ is $\{x \in G \mid xg = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

We proved that the center $Z(G)$ is a subgroup of $G$ in Exercise 21 of Section 3.2. Let $g \in G$ and $a \in Z(G)$. Then $gag^{-1} = agg^{-1} = a \in Z(G)$ and so $Z(G)$ is normal.

15. Prove that the intersection of two normal subgroups is a normal subgroup.

Let $H$ and $K$ be normal subgroups of $G$. Since $e \in H$ and $e \in K$ we have $e \in H \cap K$ and so $H \cap K \neq \emptyset$.

Let $a, b \in H \cap K$. Then $a, b \in H$ and so $ab^{-1} \in H$. Also $a, b \in K$ and so $ab^{-1} \in K$. Thus $ab^{-1} \in H \cap K$, and $H \cap K$ is a subgroup of $G$. Let $g \in G$ and $x \in H \cap K$. Then $x \in H$, so $gxg^{-1} \in H$ since $H$ is normal, and $x \in K$, so $gxg^{-1} \in K$ since $K$ is normal. Thus $gxg^{-1} \in H \cap K$ and so $H \cap K$ is a normal subgroup of $G$. 