10. Let $N$ be a normal subgroup of index $m$ in $G$. Show that $a^m \in N$ for all $a \in G$.

Let $a \in G$. Since $G/N$ has order $m$ we have $o(aN)m$ and so $(aN)^m = N$. Hence $a^mN = N$ and $a^m \in N$.

15. Find all factor groups of the dihedral group $D_4$.

We found all normal subgroups of $D_4$ in Example 3.8.9. They are $D_4, N_1 = \{e, a^2, b, a^2b\}, N_2 = \{e, a, a^2, a^3\}, N_3 = \{e, a^2, ab, a^2b\}, N = \{e, a^2\}$ and $\{e\}$. Then $D_4/D_4 \cong \langle e \rangle$; $D_4/N_1 \cong D_4/N_2 \cong D_4/N_3 \cong \mathbb{Z}_2$; and $D_4/N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by Example 3.8.11. Finally, $D_4/\langle e \rangle \cong D_4$.

17. Compute the factor group $(\mathbb{Z}_8 \times \mathbb{Z}_4)/\langle (4,2) \rangle$.

We have $\langle (2,2) \rangle = \{(0,0), (2,2), (4,0), (0,2), (2,0), (4,2)\}$. There are 4 cosets, determined by $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. A direct computation shows that each coset has order 2, so the factor group must be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

25. Give an example of a finite group $G$ with two normal subgroups $H$ and $K$ such that $G/H \cong G/K$ but $H \not\cong K$.

Let $G$ be the dihedral group $D_4$. Let $H = \{e, a, a^2, a^3\}$ and $K = \{e, a^2, b, a^2b\}$ (see Example 3.6.5). Then $H \not\cong K$ since $H$ is cyclic but $K$ is not. On the other hand, $G/H \cong G/K$ since any two groups of order 2 are isomorphic.

Another example is given by taking $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ and letting $H = \{0\} \times \mathbb{Z}_4$ and $K = \mathbb{Z}_2 \times 2\mathbb{Z}_4$.

8. For groups $G_1$ and $G_2$, determine the center of $G_1 \times G_2$.

$Z(G_1 \times G_2) = \{ (a_1, a_2) \in G_1 \times G_2 \mid (a_1, a_2)(g_1, g_2) = (g_1, g_2)(a_1, a_2) \text{ for all } (g_1, g_2) \in G_1 \times G_2 \}$

$= \{ (a_1, a_2) \in G_1 \times G_2 \mid a_1g_1 = g_1a_1 \text{ for all } g_1 \in G_1 \text{ and } a_2g_2 = g_2a_2 \text{ for all } g_2 \in G_2 \}$

$= Z(G_1) \times Z(G_2)$.

9. Show that $G/Z(G)$ cannot be a nontrivial cyclic group. (That is, if $G/Z(G)$ is cyclic, then $G$ must be abelian, and hence $Z(G) = G$.)

Suppose that $G/Z(G) = \langle aZ(G) \rangle$ is cyclic. Let $Z = Z(G)$ and let $x, y \in G$. Then $xZ = (aZ)^n$ and $yZ = (aZ)^m$ for some integers $m, n$. Hence $x = a^{n}z_1$ and $y = a^{m}z_2$ for some $z_1, z_2 \in Z$. Thus $xy = a^{n}z_1a^{m}z_2 = a^{n+m}z_1z_2 = a^{m}a^{n}z_2z_1 = yz$, and so $G$ is abelian and $G = Z(G)$. 


15. Give another proof of Theorem 7.1.1 by constructing an isomorphism from \((HN)/N\) onto \(H/(H \cap N)\).

(In our proof we constructed an isomorphism from \(H/(H \cap N)\) onto \((HN)/N\). The point is that it may be much easier to define a function in one direction than the other.)

A typical element of \(HN/N\) is of the form \(hnN\), where \(h \in H\) and \(n \in N\). Since \(nN = N\), we have \(hnN = hN\), and so each element of \(HN/N\) has the form \(hN\), where \(h \in H\).

Define \(\phi : HN/N \to H/(H \cap N)\) by \(\phi(hN) = h(H \cap N)\). To see that \(\phi\) is well-defined, suppose that \(h_1N = h_2N\), where \(h_1, h_2 \in H\). Then \(h_2^{-1}h_1 \in N\) and since \(h_2^{-1}h_1 \in H\), we have \(h_2^{-1}h_1 \in H \cap N\), and so \(h_1(H \cap N) = h_2(H \cap N)\).

Let \(h_1N, h_2N \in HN/N\). Then

\[
\phi(h_1Nh_2N) = \phi(h_1h_2(N \cap N)) = h_1(H \cap N) \cdot h_2(H \cap N) = \phi(h_1N)\phi(h_2N),
\]

and so \(\phi\) is a homomorphism.

Let \(h(H \cap N) \in H/(H \cap N)\). Then \(hN \in HN/N\) and so \(\phi(hN) = h(H \cap N)\) shows that \(\phi\) is onto.

If \(h_1N, h_2N \in HN/N\) are such that \(\phi(h_1N) = \phi(h_2N)\), then \(h_1(H \cap N) = h_2(H \cap N)\) and so \(h_2h_1^{-1} \in H \cap N \subseteq N\). Thus \(h_1N = h_2N\) and \(\phi\) is one-to-one.