MATH 154 Homework 4 - Solutions

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Hand in: AP&M 6436 (not in drop-off box)

Assigned questions to hand in:

(1) Determine the number of sequences \(\langle x_1, \ldots, x_k \rangle\) with each \(x_i \in \{1, \ldots, n\}\) with the given restriction: for \(i \leq n, x_i \neq x_{i+1}\).
Ver Part 1 pp. 16 #1 b

Solution: There are \(n\) choices for the value of \(a_1\). For \(1 < i \leq n + 1\), only \(n - 1\) options are available, namely, those in the set \(\{1, \ldots, n\}\setminus\{a_{i-1}\}\). We consider two cases, depending on whether \(n + 1 \leq k\) or \(n + 1 > k\).

- If \(n + 1 > k\), then each sequence entry after the first one has \(n - 1\) options. Therefore, there are
  \[n(n-1)^{k-1}\]
  distinct sequences.

- Otherwise, \(n + 1 \leq k\). So, the constraint that \(x_i \neq x_{i+1}\) only affects the number of choices available for sequence entries \(a_2, \ldots, a_{n+1}\). This means that for the entries \(a_{n+2}, \ldots, a_k\) there are \(n\) choices. Thus, the number of distinct sequences is
  \[n(n-1)^n n^{k-n-1} = n^{k-n}(n-1)^n\]

(2) Let \(A\) be the set of all sequences of positive integers (of any length) which add up to \(n\), and let \(B\) be the set of all subsets of \(\{1, \ldots, n-1\}\). Find a bijection \(f : A \to B\). Deduce that \(|A| = 2^{n-1}\) for \(n \geq 1\).
Ver Part 1 pp. 16 # 7

Solution: Define \(f : A \to B\) by
  \[f(\langle a_1, \ldots, a_k \rangle) = \{a_1 + \cdots + a_j : j < k\}\]

We will prove this is a well-defined bijection.

- Well-defined: we need to show that \(f(\langle a_1, \ldots, a_k \rangle) \subseteq \{1, \ldots, n-1\}\) whenever \(a_1 + \cdots + a_k = n\). In this case, each partial sum \(a_1 + \cdots + a_j\) is strictly smaller than the full sum \(a_1 + \cdots + a_k\) because each \(a_i\) is positive. Therefore, each partial sum is less than or equal to \(n - 1\) and we are done.

- Injective: Suppose \(f(\langle a_1, \ldots, a_k \rangle) = f(\langle b_1, \ldots, b_\ell \rangle)\) for some values of \(a_i, k, b_i, \ell\). We need to show that \(k = \ell\) and that for each \(i \leq k, a_i = b_i\). Note that \(|f(\langle a_1, \ldots, a_k \rangle)| = k\) because each partial sum is distinct. Similarly, \(|f(\langle b_1, \ldots, b_\ell \rangle)| = \ell\). But, by assumption, these sets are equal and so must have the same size. Thus, \(k = \ell\). Moreover, since the sets are equal, their least elements must agree. By definition, the least element of \(f(\langle a_1, \ldots, a_k \rangle)\) is \(a_1\) and the least element of \(f(\langle b_1, \ldots, b_\ell \rangle)\) is \(b_1\). Thus, \(a_1 = b_1\). Proceeding to the second least element, etc. we see that \(a_i = b_i\) for each \(i < k = \ell\). Finally,

  \[a_k = n - a_1 - \cdots - a_{k-1} = n - b_1 - \cdots - b_{\ell-1} = b_\ell\]

Thus, \(\langle a_1, \ldots, a_k \rangle = \langle b_1, \ldots, b_\ell \rangle\).
• Surjective: Let $X \subseteq \{1, \ldots, n-1\}$. If $X = \emptyset$ note that $f(\langle n \rangle) = \{a_1 + \cdots + a_j : j < 1\} = \emptyset = X$. Otherwise, let $k = |X| > 0$ and write $X$ in increasing order $x_1, \ldots, x_k$. Consider the ordered sequence

$$\langle x_1, x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}, n - x_k \rangle.$$ 

Then

$$f(\langle x_1, x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1}, n - x_k \rangle)$$

$$= \{x_1, x_1 + (x_2 - x_1), \ldots, x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_k - x_{k-1})\}$$

$$= \{x_1, x_2, x_3, \ldots, x_k\} = X,$$

as required.

Since $A$ and $B$ are in bijective correspondence, they have the same size. Moreover, the set of subsets of $\{1, \ldots, n-1\}$ has size $2^{n-1}$. Therefore, $|A| = 2^{n-1}$ as well.

(3) Find the smallest value of $m$ so that the following statement is valid: Any collection of $m$ distinct positive integers must contain at least two numbers whose sum or difference is a multiple of 10. Prove that your value is best possible.

$HHM$ 2.4.5 p. 156

Solution: We will prove that $m = 7$. First note that for $n \leq 6$, we can find $a_1, \ldots, a_n$ of distinct positive integers such that no pair of these numbers have sum or difference that is a multiple of 10. It is sufficient to consider $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 5, a_6 = 10$.

Now, we prove that if $x_1, \ldots, x_7$ are distinct positive integers then at least two of them must have sum or difference which is a multiple of 10. Consider the set of sets

$$A = \{\{0\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}\}.$$ 

Define the function $f : \mathbb{N} \to A$ where $f(x)$ is the set in $A$ containing the remainder when dividing $x$ by 10. For example, $f(14) = \{4, 6\}$. Since $|A| = 6$ and there are seven entries in the sequence $x_1, \ldots, x_7$, by the Pigeonhole Principle, there must be $i, j \in \{1, \ldots, 7\}$ such that $f(x_i) = f(x_j)$. We have two cases:

• The remainder when dividing $x_i$ and $x_j$ by 10 is the same. Then $x_i - x_j$ is a multiple of 10.

• The remainder when dividing $x_i$ and $x_j$ by 10 is different. Let $r_i, r_j$ be these remainders. By definition of the set $A$, each set in $A$ contains numbers that add up to 10. Since $f(x_i) = f(x_j)$, we have that $r_i + r_j = 10$. Therefore, $x_i + x_j$ is a multiple of 10.

(4) A noted vexillologist tells you that 30 of the 50 U.S. state flags have blue as a background color, twelve have stripes, 26 exhibit a plant or animal, nine have both blue in the background and stripes, 23 have both blue in the background and feature a plant or animal, and three have both stripes and a plant or animal. One of the flags in this last category (California) does not have any blue in the background. How many state flags have no blue in the background, no stripes, and no plant or animal featured?

$HHM$ 2.5.1 p. 161

Solution: Let $B$ be the set of flags with blue as a background color, $S$ be the set of flags with stripes, and $P$ be the set of flags with a plant or animal. Then we are told that

$$|B| = 30 \quad |S| = 12 \quad |P| = 26$$
Adding these terms up, we get

\[ |B \cap S| = 9 \quad |B \cap P| = 23 \quad |S \cap P| = 3 \quad |S \cap P \cap B| = 2. \]

We want to find \(|B^c \cap S^c \cap P^c| = |(B \cup S \cup P)^c| = 50 - |B \cup S \cup P|\). By the IEP,

\[
|B \cup S \cup P| = |B| + |S| + |P| - |B \cap S| - |B \cap P| - |P \cap S| + |B \cap S \cap P|
\]
\[ = 30 + 12 + 26 - 9 - 23 - 3 + 2 = 35. \]

(5) Use Theorem 2.6 (Inclusion-Exclusion Principle) to determine the chromatic polynomial for the yield sign (add a single edge to the bipartite graph \(K_{1, 3}\)). \(HHM\ 2.5.10a\ p. 163\)

**Solution:** For the yield sign, label the vertices and edges as follows

Let \(A_i = \{\text{colorings with } x \text{ colors where the endpoints of } e_i \text{ have same color}\}\). Then, by IEP, the chromatic polynomial of this graph is

\[
x^4 - |A_1 \cup A_2 \cup A_3 \cup A_4|
\]
\[= x^4 - |A_1| - |A_2| - |A_3| - |A_4|
+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|
- |A_1 \cap A_2 \cap A_3| - |A_1 \cap A_2 \cap A_4| - |A_1 \cap A_3 \cap A_4| - |A_2 \cap A_3 \cap A_4|
+ |A_1 \cap A_2 \cap A_3 \cap A_4|.
\]

Moreover, we compute

\[|A_1| = |A_2| = |A_3| = |A_4| = x^3\]

because have three independent choices of colors for the vertices,

\[|A_1 \cap A_2| = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3| = x^2\]

and all the rest of the terms correspond to all vertices being colored by the same color

\[|A_1 \cap A_2 \cap A_3| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = |A_1 \cap A_2 \cap A_3 \cap A_4| = x.\]

Adding these terms up, we get

\[c_{\text{yield}}(x) = x^4 - 4x^3 + 5x^2 - 2x.\]

We can also compute the polynomial directly: there are \(x\) choices of colors for \(v_2\). Once its color has been picked we have \(x - 1\) choices for \(v_1\) and \(v_3\), and \(x - 2\) choices for \(v_4\). Therefore,

\[c_{\text{yield}}(x) = x(x - 1)^2(x - 2) = (x^2 - 2x)(x^2 - 2x + 1) = x^4 - 2x^3 - 2x^3 + 4x^2 + x^2 - 2x\]
\[= x^4 - 4x^3 + 5x^2 - 2x.\]