Earlier, we defined functions in general (Eccles chapter 8). An important special case consists of functions on the reals, \( f : \mathbb{R} \to \mathbb{R} \).

**Example.**

- Polynomials, exponentials, trigonometric functions
- Modulus function
  \[
  |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
  \end{cases}
  \]
- Step function
  \[
  H(x) = \begin{cases} 
  0 & \text{if } x < 0 \\
  1 & \text{if } x \geq 0 
  \end{cases}
  \]
- Floor and ceiling functions
  \[
  \lfloor x \rfloor = \text{largest integer less than } x \quad \lceil x \rceil = \text{smallest integer greater than } x
  \]

**Definition.** Let \( f : \mathbb{R} \to \mathbb{R} \) and \( x_0 \in X \). Then \( f \) has a limit \( L \) at \( x_0 \) if for each \( \epsilon \in \mathbb{R}^+ \), there is a \( \delta \in \mathbb{R}^+ \) such that for all \( x \in \mathbb{R} \),
\[
0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon.
\]

**Example.** The function given by \( f(x) = x^2 \) has a limit 0 at \( x_0 = 0 \).

**Proof.** Let \( \epsilon \in \mathbb{R}^+ \). We want to find \( \delta \) such that for each \( x \) with \( 0 < |x| < \delta \), \( 0 < |x^2| < \epsilon \). Define \( \delta = \sqrt{\epsilon} \). Then suppose \( 0 < |x| < \delta \).
\[
|x^2| = x^2 < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon.
\]

**Example.** The modulus function has a limit 0 at \( x_0 = 0 \).

**Proof.** Let \( \epsilon \in \mathbb{R}^+ \). We want to find \( \delta \) such that for each \( x \) with \( 0 < |x| < \delta \), \( 0 < |x| - 0| < \epsilon \). Define \( \delta = \epsilon \). Then suppose \( 0 < |x| < \delta \).
\[
||x| - 0| = |x| < \delta = \epsilon.
\]

**Example.** The function given by \( f(x) = \begin{cases} 
\frac{|x|}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases} \) does not have a limit at 0.

**Proof.** Notice that if \( x < 0 \) then \( f(x) = -1 \) and if \( x > 0 \) then \( f(x) = 1 \). We will show that for any \( L \in \mathbb{R} \), \( L \) is not the limit of \( f \) at \( x_0 = 0 \). Let \( L \) be any number and let \( \epsilon = 1 \).

- Case 1: \( L \geq 0 \). Suppose \( \delta \) is a positive number. Pick some negative number \( x \) such that \( 0 < |x| < \delta \). For this \( x \), \( f(x) = -1 \) so
  \[
  |f(x) - L| = |-1 - L| = |-(L + 1)| = L + 1 \geq \epsilon.
  \]
- Case 2: \( L < 0 \). Suppose \( \delta \) is a positive number. Pick some positive number \( 0 < x < \delta \). Then \( f(x) = 1 \) and
  \[
  |f(x) - L| = |1 - L| \geq 1 = \epsilon.
  \]
Thus, in either case, there is some \( x \) in the \( \delta \)-neighbourhood of \( x_0 \) whose function value is too far away from \( L \).

\[\square\]

**Definition.** Let \( f : \mathbb{R} \to \mathbb{R}, x_0 \in \mathbb{R} \). Then \( f \) is **continuous** at \( x_0 \) if for each \( \epsilon \in \mathbb{R}^+ \), there is a \( \delta \in \mathbb{R}^+ \) such that if \( x \in \mathbb{R} \) and \( |x - x_0| < \delta \) then \( |f(x) - f(x_0)| < \epsilon \). In symbols:

\[\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in \mathbb{R} (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon).\]

We say that \( f \) is **continuous** if it is continuous at each point in its domain.

**Example.** Prove that the modulus function is continuous at 0.

**Proof.** Let \( \epsilon \in \mathbb{R}^+ \). Define \( \delta = \epsilon \). Then, if \( x \in \mathbb{R} \) and \( |x - 0| < \delta \), we are in one of two cases:

- Case 1: \( x \geq 0 \). Then \( 0 < x < \delta \). In this case,
  \[|f(x) - f(0)| = |x - 0| = |x| < \delta = \epsilon,\]
  as required.
- Case 2: \( x < 0 \). Then \( -\delta < x < 0 \). In this case,
  \[|f(x) - f(0)| = |-x - 0| = |-x| = |x| < \delta = \epsilon,\]
  as required again. \(\square\)

**Example.** Prove that the step function is continuous at \( x_0 = 1 \) but is not continuous at \( x_0 = 0 \).

**Proof.** To prove that \( H(x) \) is continuous at \( x_0 = 1 \), we need to prove that

\[\forall \epsilon \in \mathbb{R}^+ \exists \delta \in \mathbb{R}^+ \forall x \in \mathbb{R} (|x - 1| < \delta \implies |H(x) - 1| < \epsilon).\]

Suppose \( \epsilon \in \mathbb{R}^+ \) is given. Define \( \delta = \frac{1}{2} \) (notice that in this case, our choice of \( \delta \) doesn’t depend on \( \epsilon \)). Then for each \( x \in \mathbb{R} \), if

\[|x - 1| < \frac{1}{2} \quad \text{then} \quad \frac{1}{2} < x < \frac{3}{2}\]

so \( x \) is guaranteed to be positive. Therefore, \( H(x) = 1 \). That is,

\[|H(x) - 1| = |1 - 1| = 0 < \epsilon.\]

The second part asks us to prove that \( H(x) \) is not continuous at \( x_0 = 0 \). So, we need to prove that

\[\exists \epsilon \in \mathbb{R}^+ \forall \delta \in \mathbb{R}^+ \exists x \in \mathbb{R} (|x - 0| < \delta \text{ and } |H(x) - 1| \geq \epsilon)\]

We get to choose \( \epsilon \), so choose \( \epsilon = \frac{1}{2} \). Given \( \delta > 0 \), let \( x = -\frac{\delta}{2} \).

**Why?** We want \( x \) to be very close to 0 but negative because \( H(x) \) acts differently on negative numbers from how it behaves at \( x_0 = 0 \).

Then

\[|x - 0| = \left| -\frac{\delta}{2} - 0 \right| = \frac{\delta}{2} < \delta\]

and

\[|H(x) - 1| = |0 - 1| = 1 > \frac{1}{2} = \epsilon.\]

\[\square\]
The following theorem is often stated without proof in calculus classes. We now have all the ingredients to prove it. (Note: however, that the implication (b) $\Rightarrow$ (c) is a little tricky.)

**Theorem.** Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$. Then TFAE ("the following are equivalent")

(a) $f$ is continuous at $x_0$.
(b) If $\langle x_n \rangle$ is a sequence in $\mathbb{R}$ that converges to $x_0$ then the sequence $\langle f(x_n) \rangle$ converges to $f(x_0)$.
(c) $f$ has a limit at $x_0$ and $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

**Proof.** Prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Suppose $f$ is continuous at $x_0$ and $\langle x_n \rangle$ is a sequence that converges to $x_0$. Let $\epsilon > 0$. Since $f$ is continuous at $x_0$, there is $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \epsilon.$$ 

But $\lim_{n \rightarrow \infty} x_n = x_0$ so there is $N$ such that for all $n \geq N$

$$|x_n - x_0| < \delta.$$ 

Thus, for this $N$, for all $n \geq N$,

$$|f(x_n) - f(x_0)| < \epsilon.$$ 

(b) $\Rightarrow$ (c) We prove the contrapositive. Namely, we will prove that if there is $\epsilon > 0$ such that for all $\delta > 0$, there is some $x$ such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$.

then

there is a sequence which converges to $x_0$ but whose image sequence does not converge to $f(x_0)$.

So assume there is $\epsilon_0 > 0$ such that for all $\delta > 0$, there is some $x$ such that $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \epsilon$. In particular, we consider \( \delta_1 = \frac{1}{2}, \delta_2 = \frac{1}{4}, \delta_3 = \frac{1}{8}, \ldots, \) and in general, \( \delta_n = \frac{1}{2^n}. \) Since each $\delta_n > 0$, the assumption guarantees that there is some number, call it $x_n$, such that

$$|x_n - x_0| < \delta_n = \frac{1}{2^n} \quad \text{and} \quad |f(x_n) - f(x_0)| \geq \epsilon_0.$$ 

Now, consider the sequence $\langle x_n \rangle$. First, we prove that $\lim_{n \rightarrow \infty} x_n = x_0$. To do so, we need to show that

$$\forall \epsilon \in \mathbb{R}^+ \exists N \in \mathbb{Z}^+ \forall n \in \mathbb{Z}^+(n \geq N \Rightarrow |x_n - x_0| < \epsilon).$$ 

Given some $\epsilon \in \mathbb{R}^+$, define $N = \lceil -\log_2(\epsilon) \rceil + 1$. Then, if $n \geq N$

$$|x_n - x_0| \leq 2^{-n} \leq 2^{-N} < 2^{\log_2(\epsilon)} = \epsilon.$$ 

But, we also prove that $\lim_{n \rightarrow \infty} f(x_n) \neq f(x_0)$. We need to show that

$$\exists \epsilon \in \mathbb{R}^+ \forall N \in \mathbb{Z}^+ \exists n \in \mathbb{Z}^+(n \geq N \text{ and } |f(x_n) - f(x_0)| \geq \epsilon).$$ 

The witness will be $\epsilon = \epsilon_0$ from the beginning of this proof because we defined it to be such that all $f(x_n)$ are at least $\epsilon_0$ away from $f(x_0)$. Formally, if $N \in \mathbb{Z}^+$, let $n = N$ and notice that

$$n \geq N \quad \text{and} \quad |f(x_n) - f(x_0)| \geq \epsilon_0.$$
Thus, we have shown that it is not the case that if \( \langle x_n \rangle \) is a sequence in \( \mathbb{R} \) that converges to \( x_0 \) then the sequence \( \langle f(x_n) \rangle \) converges to \( f(x_0) \). In other words, the proof of the contrapositive is complete.

(c) \( \implies \) (a) Suppose \( \lim_{x \to x_0} f(x) = f(x_0) \). Let \( \epsilon > 0 \). By definition of limit of a function, there is \( \delta > 0 \) such that

\[
0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.
\]

This is part of the requirement in the definition of continuity; it remains to consider the case \( |x - x_0| = 0 \). In this case, \( x = x_0 \) and \( f(x) - f(x_0) = 0 \).

\( \square \)

Note: the above can be made to work with functions whose domains are subsets of \( \mathbb{R} \) but then need to worry about accumulation points.