#7.2 (i) Counterexample: This is the statement that if \( m \in \mathbb{Z}^+ \) then

\[
\{ n \in \mathbb{Z}^+ : m \leq n \} = \mathbb{Z}^+.
\]

When \( m = 2 \), we have \( 1 \notin \{ n \in \mathbb{Z}^+ : 2 \leq n \} \) so the set does not equal \( \mathbb{Z}^+ \).

(iii) Proof. This is the statement that if \( m \in \mathbb{Z}^+ \) then the set \( \{ n \in \mathbb{Z}^+ : m \leq n \} \) is nonempty. Since \( m \in \{ n \in \mathbb{Z}^+ : m \leq n \} \), we see that the statement is true for all integers \( m \in \mathbb{Z}^+ \).

(v) Proof. This is the statement that if \( n \in \mathbb{Z}^+ \) then the set \( \{ m \in \mathbb{Z}^+ : m \leq n \} \) is nonempty. Since \( n \in \{ m \in \mathbb{Z}^+ : m \leq n \} \), we see that the statement is true for all integers \( n \in \mathbb{Z}^+ \).

#7.4 (iii) Proof. This is the statement that if \( x \in \mathbb{R} \) then the set \( \{ y \in \mathbb{R} : xy = 0 \} \) is nonempty. Since \( x \cdot 0 = 0 \), we see that \( 0 \in \{ y \in \mathbb{R} : xy = 0 \} \), and hence the set is nonempty.

(iv) Proof. This is the statement that the set \( \{ y \in \mathbb{R} : \forall x \in \mathbb{R}, xy = 0 \} \) is nonempty. Since \( x \cdot 0 = 0 \), we see that \( 0 \in \{ y \in \mathbb{R} : \forall x \in \mathbb{R}, xy = 0 \} \), and hence the set is nonempty.

#8.1 Proof. To show that \( g \) is well-defined, we need to show that to each \( (x, y) \in \mathbb{R}^2 \), the function \( g \) assigns a unique real number. Let \( (x, y) \in \mathbb{R} \). By trichotomy (Axiom 3.1.2 (i)), we have that exactly one of the three possibilities \( x < y \), \( x = y \), \( x > y \) is true.

Case 1: If \( x < y \), then \( x \leq y \) and \( x \neq y \) so \( g(x, y) = y \) is well-defined.

Case 2: If \( x = y \) then \( x \leq y \) so \( g(x, y) = x \). We also have, however, that \( x \geq y \) so \( g(x, y) = y \). Since \( x = y \), we see that \( g \) is well-defined in this case because \( g(x, y) = x = y \).

Case 3: If \( x > y \), then \( x \geq y \) and \( x \neq y \) so \( g(x, y) = x \) is well-defined. This concludes the proof that \( g \) is well-defined.

To show that \( g = f \), we must show that for each \( (x, y) \in \mathbb{R}^2 \) we have \( g(x, y) = f(x, y) \). Let \( (x, y) \in \mathbb{R} \). By trichotomy (Axiom 3.1.2 (i)), we have that exactly one of the three possibilities \( x < y \), \( x = y \), \( x > y \) is true.
Case 1: If $x < y$, then $|x - y| = y - x$ so

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y + y - x}{2} = y = g(x, y)$$

Case 2: If $x = y$ then

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y}{2} + 0 = \frac{x + x}{2} = x = g(x, y).$$

Case 3: If $x > y$, then $|x - y| = x - y$ so

$$f(x, y) = \frac{x + y}{2} + \frac{|x - y|}{2} = \frac{x + y + x - y}{2} = x = g(x, y).$$

Hence, we have shown that $g = f$.

8.4 Proof. We are required to prove that for all $\epsilon \in \mathbb{R}^+$, there exists $N \in \mathbb{Z}^+$ such that for all $n \in \mathbb{Z}^+$ with $n \geq N$ we have $1/n < \epsilon$. Given a positive real $\epsilon \in \mathbb{R}^+$, we have $1/n < \epsilon$ if and only if $n > 1/\epsilon$. Hence, if we choose $N > 1/\epsilon$ then $n \geq N > 1/\epsilon$ implies $1/n < \epsilon$ as desired.

Continuity (a): Proof. We are required to show that for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x - 0| < \delta$, then $|f(x) - 1| < \epsilon$. Given $\epsilon > 0$, let $\delta = \min\{\epsilon, 1\}$. Since $\delta \leq 1$, we have $x \neq 1$, and thus

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \Rightarrow |f(x) - 1| = |(x + 1) - 1| = |x - 0| < \min\{\epsilon, 1\} \leq \epsilon.$$

Continuity (b): Proof. To show that $f$ is not continuous at $x = 0$, we must show that there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $0 < |x - 0| < \delta$ and $|f(x) - f(0)| \geq \epsilon$. Let $\epsilon = 1/2$. We will show that for all $\delta > 0$, there exists an $x \in \mathbb{R}$ such that $|x| < \delta$ and $|f(x)| > 1/2$. In part (a), we showed that $\lim_{x \to 0} = 1$. This means that there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x| < \delta$ we have $|f(x) - 1| < 1/4$; in other words all $x \in \mathbb{R}$ with $|x| < \delta$ satisfy $3/4 < f(x) < 5/4$. Consequently, for any $\delta > 0$, if we choose $x \in \mathbb{R}$ such that $|x| < \min\{\delta, \delta\}$ then $|x| < \delta$ and $|f(x)| > 1/2$. 

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