(Short question) Prove that $A_L \vdash \forall x \forall y (x < y \rightarrow x \neq y)$.

(Long question) Let $\varphi$ be the wff

$\forall x (S^2 x \neq y) \rightarrow \exists x (Sy = x \land z = S^3 x)$.

Find a quantifier-free formula $\psi$ such that $\models _{\mathfrak{S}} \psi \iff \varphi$. Use $\psi$ to find the binary relation defined in $\mathfrak{S}$ by $\varphi$. *Hint: start with the prenex normal form of $\varphi$.*
• Recall that \( L4 \) (antisymmetry) is \( \forall x\forall y(x < y \rightarrow \neg y < x) \). On the other hand, axiom group 6 for first-order deduction includes \( x = y \rightarrow (\alpha \rightarrow \alpha') \) for any \( \alpha \) where \( \alpha' \) is \( \alpha \) with some \( x \)'s replaced by \( y \)'s. If we consider \( \alpha \) to be \( x < y \) then this axiom implies that \( x = y \rightarrow (x < y \rightarrow x < x) \). Combining this with \( x < x \rightarrow \neg x < x \) (instantiating \( L4 \)) we can prove that \( x < y \rightarrow x \neq y \) via the tautology

\[
(A \rightarrow B \rightarrow C) \rightarrow (B \rightarrow \neg C) \rightarrow (B \rightarrow \neg A).
\]

To make this formal, we can use the Generalization Theorem and Rule T.

Alternatively, by Soundness and Completeness, proving that \( A_L \models \forall x\forall y(x < y \rightarrow x \neq y) \) is equivalent to proving that \( A_L \models \forall x\forall y(x < y \rightarrow x \neq y) \). Let \( \mathfrak{A} \) be model for \( A_L \) and let \( a, b \in |\mathfrak{A}| \). We must prove that \( \models_{\mathfrak{A}} (x < y \rightarrow x \neq y) \). By definition of satisfaction, this is exactly the fact that \( a \neq^\mathfrak{A} b \) or \( a \neq b \). Suppose (towards a contradiction) that \( a \neq^\mathfrak{A} b \) and \( a = b \). Since \( a = b \), these two are names of the same object. Namely, because we have that \( a <^\mathfrak{A} b \) we may conclude that \( a <^\mathfrak{A} a \). Since \( \mathfrak{A} \) is a model of \( A_L \), and in particular, of \( L4 \), we have \( \models_{\mathfrak{A}} \forall x\forall y(x < y \rightarrow \neg y < x) \). Instantiating \( x, y \) as \( a: a \neq^\mathfrak{A} a \). This is a contradiction, so we are done.

• To put the formula in prenex normal form, we rename the first quantified variable

\[
\forall w(S^2w \neq y) \rightarrow \exists x(Sy = x \land z = S^3x)
\]

then use the templates for dealing with \( \rightarrow \):

\[
\exists w\exists x(S^2w \neq y \rightarrow (Sy = x \land z = S^3x)).
\]

We rewrite the quantifier-free part in DNF (where \( A \rightarrow (B \land C) \models_{\mathfrak{A}} \neg A \lor (B \land C) \))

\[
\exists w\exists x(S^2w = y \lor (Sy = x \land z = S^3x)).
\]

This wff is logically equivalent to

\[
\exists x\exists w(S^2w = y) \lor \exists w\exists x(Sy = x \land z = S^3x).
\]

We work on each disjunct in turn. For the first, note that \( x \) is not mentioned.

\[
\models_{\mathfrak{A}} \exists w(S^2w = y) \leftrightarrow (0 \neq y \land 1 \neq y).
\]

For the second disjunct \( w \) is not mentioned:

\[
\models_{\mathfrak{A}} \exists x(Sy = x \land z = S^3x) \leftrightarrow (z = S^4y).
\]

Putting these together:

\[
\models_{\mathfrak{A}} (\forall w(S^2w \neq y) \rightarrow \exists x(Sy = x \land z = S^3x)) \leftrightarrow ((0 \neq y \land 1 \neq y) \lor (z = S^4y))
\]

Therefore, the binary relation defined by the wff in \( \mathfrak{M}_S \) is

\[
\{(a, b) \in \mathbb{N}^2 : \models_{\mathfrak{A}} ((0 \neq y \land 1 \neq y) \lor (z = S^4y))[[a, b]] = \{(a, b) : a \geq 2, b \in \mathbb{N}\} \cup \{(a, a + 4) : a \in \mathbb{N}\}.
\]