(a) Prove that
\[ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{D} \quad \varepsilon_1 \neq \varepsilon_2 \]
for \( \varepsilon_1, \varepsilon_2 \) nonempty expressions.

(b) Prove that
\[ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{D} \quad \varepsilon_1 \neq \varepsilon_2 \]
for \( \varepsilon_1, \varepsilon_2 \) nonempty expressions.

Note that it follows that if \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) are all nonempty then \( \varepsilon_i \neq \varepsilon_1 \varepsilon_2 \varepsilon_3 \) for \( i = 1, 2, 3 \).
(a) By definition of Gödel numbers,
\[ \Psi(\varepsilon_1 \varepsilon_2) = \Psi(\varepsilon_1) \ast \Psi(\varepsilon_2). \]
Therefore, it suffices to prove that for any two sequence numbers \( a_1, a_2 \) neither of which is 1 (the image of the empty sequence), \( a_1 + 1 < a_1 \ast a_2 \). But,
\[ a_1 \ast a_2 = a_1 \cdot \prod_{i < \text{lh} \ a_2} p^{(a_2)_i+1}_{i+\text{lh} \ a_1} \]
and the product is a nonempty (since \( \text{lh} \ a_2 > 0 \)) product of terms each of which is greater than or equal to 2, so
\[ a_1 \ast a_2 = a_1 \cdot \prod_{i < \text{lh} \ a_2} p^{(a_2)_i+1}_{i+\text{lh} \ a_1} \geq 2a_1 = a_1 + a_1 > a_1 + 1. \]

(b) Similarly, it suffices to prove that for any two sequence numbers \( a_1, a_2 \) neither of which is 1 (the image of the empty sequence), \( a_2 + 1 < a_1 \ast a_2 \). But,
\[ a_1 \ast a_2 = a_1 \cdot \prod_{i < \text{lh} \ a_2} p^{(a_2)_i+1}_{i+\text{lh} \ a_1} > a_1 \cdot \prod_{i < \text{lh} \ a_2} p^{(a_2)_i+1}_{i} \]
since \( \text{lh} \ a_1 > 0 \) and the primes are listed in increasing order. But now, this product is exactly \( a_2 \) so
\[ a_1 \ast a_2 > a_1 \cdot a_2 \geq 2a_2 = a_2 + a_2 > a_2 + 1. \]