1. (10 points) Let \( m, n \in \mathbb{Z} \). Show that the rings \( (\mathbb{Z} \oplus \mathbb{Z})/((m) \oplus (n)) \) and \( \mathbb{Z}_m \oplus \mathbb{Z}_n \) are ring isomorphic.

Let \( \Phi : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}_m \oplus \mathbb{Z}_n \)

\( \Phi(a, b) = (a \mod m, b \mod n) \)

1. \( \Phi(a + az, b + bz) = (a + az \mod m, b + bz \mod n) = (a \mod m, b \mod n) + (az \mod m, bz \mod n) = \Phi(a, b) + \Phi(az, bz) \)

2. \( \Phi(a + az, b + bz) = (a + az \mod m, b + bz \mod n) = (a \mod m, b \mod n) = \Phi(a, b) + \Phi(az, bz) \)

Thus \( \Phi \) is a ring hom.

2. \( \Phi \) is onto if \( (a, b) \in \mathbb{Z}_m \oplus \mathbb{Z}_n \), \( \Phi(a, b) = (a, b) \).

3. \( \text{Ker} \Phi = (m \oplus n) \)

\( \Phi(a, b) = (0, 0) \) \( \iff \) \( a = 0 \mod m \)

\( b = 0 \mod n \)

(\( \implies \) \( a \in (m) \) \& \( b \in (n) \))

(\( \implies \) \( (a, b) \in (m \oplus n) \))

Thus, by the 1st iso. Thm, \( \mathbb{Z} \oplus \mathbb{Z} / ((m) \oplus (n)) \cong \mathbb{Z}_m \oplus \mathbb{Z}_n \)
2. (10 points) Let $R$ and $S$ be commutative rings with unity and $\phi : R \to S$ a ring homomorphism onto $S$. If the characteristic of $R$ is non-zero, show that the characteristic of $S$ divides the characteristic of $R$.

By Thm, $\phi(1) \neq 1_S$ since $\phi$ is onto.

By Thm, $\text{char}(R) = \text{ least integer } n \text{ s.t. } n \cdot 1_R = 0$

similar for $S$.

Let $n = \text{char}(R)$. Then $n \cdot 1_R = 0$, since $\phi(0) = 0$.

Thus $0 = \phi(n \cdot 1_R) = \phi(1_R + \ldots + 1_R)$

$= \phi(1_R) + \ldots + \phi(1_R)$

$= n \cdot \phi(1_R)$

$= n \cdot 1_S$

Thus $\text{char}(S) \leq n$. Let $m = \text{char}(S)$. Show $m | n$.

Since $m \leq n$, $n = m \cdot q + r$, $0 \leq r < m$ by division for integers.

Then $0 = n \cdot 1_S = (m \cdot q + r) \cdot 1_S = q \cdot (m \cdot 1_S) + r \cdot 1_S$

$= r \cdot 1_S$ since $m \cdot 1 = 0$

But $m$ is least s.t. $m \cdot 1_S = 0$ & $r < m$, thus $r = 0$.

Thus $n = m \cdot q \Rightarrow m | n \Rightarrow \text{char}(S) | \text{char}(R)$.

$\therefore (m \cdot q) \cdot 1 = 1 \cdot \ldots \cdot 1 = (1 + \ldots + 1) + \ldots + (1 + \ldots + 1) \in \text{char}_S$

$= q \cdot (m \cdot 1)$
3. (10 points) Let $R$ and $S$ be commutative rings with unity and $\phi : R \to S$ a ring homomorphism onto $S$. Show that if $I$ is a maximal ideal of $S$, then $\phi^{-1}(I)$ is a maximal ideal in $R$.

By Thm, $\phi^{-1}(I)$ is an ideal.

Let $\phi^{-1}(I) = J \subseteq R$, $J$ an ideal.

Since $\phi$ is onto, by Thm, $\phi(J)$ is an ideal in $S$.

Thus $I \subseteq \phi(J) \subseteq S$.

But $I$ is maximal in $S$, thus $\phi(J) = S$.

Thus $J = \phi^{-1}(\phi(J)) = \phi^{-1}(S) = R$ since $\phi$ is onto.

Thus $\phi^{-1}(I)$ is a maximal ideal.
4. (10 points) Let $F$ be a field with $a, b \in F$, $a \neq b$ and $f(x) \in F[x]$. Suppose

$$f(x) = (x - a)^k q(x)$$

where $q(x) \in F[x]$ and $b$ is a zero of $q(x)$. Show that the multiplicity of $b$ as a zero of $q(x)$ is equal to the multiplicity of $b$ as a zero of $f(x)$.

Let $q(x) = (x - b)^j h(x)$ where $h(b) \neq 0$. Then the multiplicity of $b$ as a zero of $q(x)$ is $j$.

$$f(x) = (x - a)^k (x - b)^j h(x)$$

$$= (x - b)^j g(x), \quad g(x) = (x - a)^k h(x).$$

Moreover $g(b) = (b - a)^k h(b) \neq 0$.

Since $b \neq a$, $h(b) \neq 0$, $F$ is a field.

Thus, by definition, the multiplicity of $b$ as a zero of $f(x)$ is $j$. 
5. (10 points) Prove that the ideal \((x)\) is prime in \(\mathbb{Q}[x]\).

Let \(fg(x) \in \mathbb{Q}[x]\) s.t. \(f(x) \cdot g(x) \in \langle x \rangle\).

Show \(f(x) \in \langle x \rangle\) or \(g(x) \in \langle x \rangle\).

By the division algorithm,

\[
f(x) = x \cdot q_1(x) + r_1 \quad \text{where } \deg(r_1) \leq 0 \quad (\text{i.e. } r_1 \text{ is a const})
\]

\[
g(x) = x \cdot q_2(x) + r_2 \quad \text{r_2 a const.}
\]

\[
f(x)g(x) = (xq_1(x) + r_1)(xq_2(x) + r_2)
\]

\[
= x^2q_1(x)q_2(x) + x(q_1(x)r_2) + x(q_2(x)r_1) + r_1r_2
\]

\[
= x(q_1(x)q_2(x) + q_1(x)r_2 + q_2(x)r_1) + r_1r_2
\]

But \(f(x)g(x) \in \langle x \rangle\), thus \(r_1r_2 = 0\).

Since \(\mathbb{Q}\) is a field (thus an integral domain),

\[
\Rightarrow r_1 = 0 \quad \text{or} \quad r_2 = 0
\]

\[
\Rightarrow f(x) = xq_1(x) \quad \text{or} \quad g(x) = xq_2(x)
\]

\[
\Rightarrow f(x) \in \langle x \rangle \quad \text{or} \quad g(x) \in \langle x \rangle.
\]

Thus, \(\langle x \rangle\) is a prime ideal in \(\mathbb{Q}[x]\).