Consider the set of polynomials with real-valued coefficients, $\mathbb{R}[x]$.

(1) What is the identity of this ring? What is the unity?

(2) Is $3x$ a zero divisor of this ring? Justify your answer.

(3) Is $3x$ a unit of this ring? Justify your answer.

(4) Is 3 a unit of this ring? Justify your answer.
(1) The (additive) identity is the zero polynomial, 0, and the unity is the constant polynomial with value 1. To prove this, consider an arbitrary polynomial \(a_n x^n + \ldots + a_1 x + a_0\) with \(a_i \in \mathbb{R}\). Then

\[
0 + (a_n x^n + \ldots + a_1 x + a_0) = a_n x^n + \ldots + a_1 x + a_0 = (a_n x^n + \ldots + a_1 x + a_0) + 0
\]

and

\[
1 \cdot (a_n x^n + \ldots + a_1 x + a_0) = a_n x^n + \ldots + a_1 x + a_0 = (a_n x^n + \ldots + a_1 x + a_0) \cdot 1.
\]

(2) No. Let \(a_n x^n + \ldots + a_0 \in \mathbb{R}[x]\) and suppose that

\[
(3x)(a_n x^n + \ldots + a_1 x + a_0) = 0.
\]

We will prove that each \(a_i = 0\). Expanding out the product:

\[
0 = 3a_n x^{n+1} + \ldots + 3a_1 x^2 + 3a_0 x.
\]

But, a polynomial is zero iff each of its coefficients is. Thus,

\[
3a_n = 0, \ldots, 3a_1 = 0, 3a_0 = 0.
\]

Dividing each of these equations by 3 (since \(\mathbb{R}\) is a field), we see that

\[
a_n = \cdots = a_1 = a_0 = 0.
\]

(3) No. Suppose, towards a contradiction that \(3x\) had a multiplicative inverse in \(\mathbb{R}[x]\), say \(b_m x^m + \cdots + b_0\). By definition,

\[
(3x)(b_m x^m + \cdots + b_0) = 1 = 1 + 0x + 0x^2 + \cdots.
\]

Expanding out:

\[
3b_m x^{m+1} + \ldots + 3b_0 x + 0 = 0x^{m+1} + \cdots + 0x + 1.
\]

Two polynomials are equal iff the coefficients each of the like terms are equal. But, the constant coefficient of the LHS is zero whereas of the RHS is 1.

(4) Yes. Its inverse is the constant polynomial \(\frac{1}{3}\).