Let $R$ be a commutative ring. Let $r \in R$. Consider the function

$$\varphi_r : R[x] \to R$$

defined by

$$\varphi(p(x)) = p(r) = \text{the result of substituting } r \text{ for } x \text{ in the polynomial } p(x).$$

Prove that $\varphi_r$ is a ring homomorphism.
We need to show that the function preserves the ring operations. Let $p(x) = a_n x^n + \cdots + a_0$, $q(x) = b_m x^m + \cdots + b_0$ be in $R[x]$. Assume, wlog, that $n \geq m$.

- **Addition:**
  \[
  \varphi_r(p(x) + q(x)) = \varphi_r(a_n x^n + \cdots + a_{m+1} x^{m+1} + (a_m + b_m) x^m + \cdots + (a_0 + b_0))
  = a_n r^n + \cdots + a_{m+1} r^{m+1} + (a_m + b_m) r^m + \cdots + (a_0 + b_0)
  \]
  Dist. in $R = a_n r^n + \cdots + a_{m+1} r^{m+1} + a_m r^m + b_m r^m + \cdots + a_0 + b_0$
  Comm. of $+ \text{ in } R = (a_n r^n + \cdots + a_{m+1} r^{m+1} + a_m r^m + \cdots + a_0 + b_0)$
  \[
  = \varphi_r(p(x)) + \varphi_r(q(x)).
  \]

- **Multiplication:**
  \[
  \varphi_r(p(x)q(x)) = \varphi_r \left( \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j} \right) = \left( \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j r^{i+j} \right)
  \]
  Comm. of $+ \text{ in } R = \sum_{j=0}^{m} \sum_{i=0}^{n} a_i b_j r^{i+j}$
  Comm. of $\cdot \text{ in } R = \sum_{j=0}^{m} \sum_{i=0}^{n} b_j a_i r^{i+j}$
  \[
  = \varphi_r(p(x)) \varphi_r(q(x)).
  \]